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**ELECTRODYNAMICS AND WAVE PROPAGATION**

# Chirped and Chirp-Free Optical Solitons in Fiber Bragg Gratings with Kudryashov's Model in Presence of Dispersive Reflectivity

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**Abstract**— In this paper, the extended Kudryashov's method has been successfully implemented to obtain optical solitons and other solutions to Kudryashov's model in fiber Bragg gratings. Dark and singular optical solitons emerge from the scheme along with their respective existence criteria.

**Keywords:** solitons, Bragg gratings, dispersive reflectivity, extended Kudryashov's method

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## 1. INTRODUCTION

The engineering marvel of optical fiber Bragg gratings (BGs) has left a lasting impression in the field of optoelectronics [1–11, 17–25]. As mentioned earlier, the newly proposed model, namely Kudryashov's equation (KE), has opened up a flood of opportunities in the field of nonlinear optics [12–16]. Here we are on yet another avenue with KE, namely its applicability to fiber BGs. This innovative technological miracle is applicable to optical fibers when the much needed delicate balance between chromatic dispersion (CD) and fiber nonlinearity is at stake if CD runs low. In such a situation, BGs artificially introduces induced dispersion that restores this balance for sustainability of soliton transmission along intercontinental distances. Today's paper, for the first time, studies fiber BGs with KE in presence of dispersive reflectivity. KE was introduced during 2019 and was later applied to study its conservation laws and also it was addressed by Lie symmetry analysis [12–16]. In the past, solitary wave solitons as well as soliton solutions were recovered by the aid of a variant form of simplest equation scheme [30–33].

BGs has also been studied with several forms of non-Kerr refractive index. It was primarily addressed with numerical schemes although lately analytical approaches have been implemented [8–11, 23–26]. Analytically, for parabolic law nonlinearity, BG has been studied using Jacobi's elliptic function [23]; for quadratic–cubic nonlinearity, sub–ODE approach was adopted [24]; for parabolic–nonlocal combo nonlinearity, three integration schemes were implemented and they are unified Riccati equation method, new extended auxiliary equation scheme and unified auxiliary equation algorithm [25]. Finally, for polynomial law, the sub–ODE approach was once again successfully applied [26]. Today's paper focuses on the study of soliton solutions with fiber BGs, having KE as its platform, in presence of dispersive reflectivity. The adopted scheme of study in this paper is extended Kudryashov's method. The details are inked in the rest of the paper after drawing a schematic pen-picture to the model.

### *1.1. Governing Model*

The governing KE with an arbitrary power of nonlinearity is given by [12–16]:

$$iq_t + aq_{xx} + \left(\frac{b_1}{|q|^{2n}} + \frac{b_2}{|q|^n} + b_3|q|^n + b_4|q|^{2n}\right)q = 0, \qquad (1)$$

with $i = \sqrt{-1}$, where the first term is the linear temporal evolution, while the coefficient of $a$ represents chromatic dispersion (CD) and the constant coefficients $b_l$ for $l = 1, 2, 3, 4$ are nonlinear and represent law of refractive index of an optical fiber that accounts for self-phase modulations (SPM). Then, the nonlinearity index $n$ is the power law parameter. The following subsections will introduce KE in fiber BGs with three cases, namely $n = 1$, $n = 2$ and $n = 3$.

One must note that for equation (1), the case when $|q| \to C_1 \neq 0$ also permits solutions that have been recently discussed [27–29]. It must also be noted that equation (1) can be rewritten as:

$$(iq_t + aq_{xx})|q|^{2n} + \left(b_1 + b_2|q|^n + b_3|q|^{3n} + b_4|q|^{4n}\right)q = 0,$$

which no longer has division by zero problem.

#### 1.1.1. CASE–I: ($n = 1$)

For optical fibers with differential group delay, KE (1) splits into two components that leads to the effect of fiber BGs at $n = 1$. Thus, the vector–coupled KE reads

$$iu_t + a_1v_{xx} + \frac{f_1u}{b_1|u|^2 + c_1|v|^2} + \frac{d_1u}{\sqrt{|u|^2 + |v|^2}} + e_1u\sqrt{|u|^2 + |v|^2} + \left(g_1|u|^2 + h_1|v|^2\right)u + i\alpha_1u_x + \beta_1v = 0, \quad (2)$$

$$iv_t + a_2u_{xx} + \frac{f_2v}{b_2|v|^2 + c_2|u|^2} + \frac{d_2v}{\sqrt{|v|^2 + |u|^2}} + e_2v\sqrt{|v|^2 + |u|^2} + \left(g_2|v|^2 + h_2|u|^2\right)v + i\alpha_2v_x + \beta_2u = 0, \quad (3)$$

where $a_j$, $f_j$, $b_j$, $c_j$, $d_j$, $e_j$, $g_j$, $h_j$, $\alpha_j$ and $\beta_j$, $(j = 1, 2)$ are constants. Here, the dependent variables $u(x,t)$ and $v(x,t)$ represent forward and backward propagating waves, respectively while the independent variables $x$ and $t$ represent spatial and temporal variables, respectively. Next, the coefficients $a_j$ for $j = 1, 2$ are the coefficients of dispersive reflectivity, while $b_j$ and $g_j$ are the coefficients of SPM and the coefficients $c_j$ and $h_j$ represent the cross–phase modulation (XPM). The coefficients $f_j$, $d_j$ and $e_j$ represent the combination of SPM and XPM. Finally, $\alpha_j$ represent inter–modal dispersion and $\beta_j$ are detuning parameters.

#### 1.1.2. CASE–II: ($n = 2$)

For optical fibers with differential group delay, KE splits into two components that leads to the effect of fiber BGs with $n = 2$. Thus, the vector–coupled KE reads:

$$iu_t + a_1v_{xx} + \frac{f_1u}{b_1|u|^4 + c_1|u|^2|v|^2 + d_1|v|^4} + \frac{e_1u}{g_1|u|^2 + h_1|v|^2} + \left(r_1|u|^2 + q_1|v|^2\right)u + \left(\xi_1|u|^4 + \zeta_1|u|^2|v|^2 + \eta_1|v|^4\right)u + i\alpha_1u_x + \beta_1v = 0, \quad (4)$$

$$iv_t + a_2u_{xx} + \frac{f_2v}{b_2|v|^4 + c_2|v|^2|u|^2 + d_2|u|^4} + \frac{e_2v}{g_2|v|^2 + h_2|u|^2} + \left(r_2|v|^2 + q_2|u|^2\right)v + \left(\xi_2|v|^4 + \zeta_2|v|^2|u|^2 + \eta_2|u|^4\right)v + i\alpha_2v_x + \beta_2u = 0, \quad (5)$$

where $a_j$, $f_j$, $b_j$, $c_j$, $d_j$, $e_j$, $g_j$, $h_j$, $r_j$, $q_j$, $\xi_j$, $\zeta_j$, $\eta_j$, $\alpha_j$ and $\beta_j$ for $j = 1, 2$ are constants. The coefficients $a_j$ stand for dispersive reflectivity, while the coefficients $b_j$, $g_j$, $r_j$ and $\xi_j$ give SPM and the coefficients $c_j$, $d_j$, $h_j$, $q_j$, $\eta_j$ and $\zeta_j$ are XPM, respectively. The coefficients $f_j$ and $e_j$ represent the combination of SPM and XPM. Finally, $\alpha_j$ represent inter-modal dispersion and $\beta_j$ are detuning parameters and $i = \sqrt{-1}$.

#### 1.1.3. CASE–III: ($n = 3$)

For optical fibers with differential group delay, KE splits into two components that leads to the effect of fiber BGs for $n = 3$. Thus, the vector–coupled KE is:

$$iu_t + a_1v_{xx} + \frac{f_1u}{b_1|u|^6 + c_1|u|^4|v|^2 + d_1|u|^2|v|^4 + e_1|v|^6} + \frac{g_1u}{\left(\xi_1|u|^2 + \zeta_1|v|^2\right)\sqrt{|u|^2 + |v|^2}} + u\left(l_1|u|^2 + m_1|v|^2\right) \times \sqrt{|u|^2 + |v|^2} + \left(q_1|u|^6 + r_1|u|^4|v|^2 + \delta_1|u|^2|v|^4 + \theta_1|v|^6\right)u + i\alpha_1u_x + \beta_1v = 0, \quad (6)$$

$$iv_t + a_2u_{xx} + \frac{f_2v}{b_2|v|^6 + c_2|v|^4|u|^2 + d_2|v|^2|u|^4 + e_2|u|^6} + \frac{g_2v}{\left(\xi_2|v|^2 + \zeta_2|u|^2\right)\sqrt{|v|^2 + |u|^2}} + v\left(l_2|v|^2 + m_2|u|^2\right) \times \sqrt{|v|^2 + |u|^2} + \left(q_2|v|^6 + r_2|v|^4|u|^2 + \delta_2|v|^2|u|^4 + \theta_2|u|^6\right)v + i\alpha_2v_x + \beta_2u = 0, \quad (7)$$

where $a_j$, $f_j$, $b_j$, $c_j$, $d_j$, $e_j$, $g_j$, $\xi_j$, $\zeta_j$, $l_j$, $m_j$, $q_j$, $r_j$, $\delta_j$, $\theta_j$, $\alpha_j$, $\delta_j$ and $\beta_j$ for $j = 1, 2$ are constants. The coefficients $a_j$ are the coefficients of dispersive reflectivity, while the coefficients $b_j$ and $q_j$ give SPM and the coefficients $c_j$, $d_j$, $e_j$, $r_j$, $\delta_j$ and $\theta_j$ are XPM, respectively. The coefficients $f_j$, $g_j$, $\xi_j$, $\zeta_j$, $l_j$ and $m_j$ represent the combination of SPM and XPM. Finally, $\alpha_j$ represent inter-modal dispersion and $\beta_j$ are detuning parameters.

## 2. MATHEMATICAL ANALYSIS (CASE–I: $n = 1$)

### *2.1. Chirp–Free Solitons*

To this aim, we introduce the transformation:

$$u(x,t) = \varphi_1(\xi)\exp[i\eta(x,t)], \quad (8)$$

$$v(x,t) = \varphi_2(\xi)\exp[i\eta(x,t)], \quad (9)$$

and

$$\xi = x - vt, \quad \eta(x,t) = -kx + \omega t + \theta_0, \quad (10)$$

where $v$, $k$, $\omega$ and $\theta_0$ are all non zero constants to be determined which represent the velocity of soliton, frequency of soliton, wave number and phase constants, respectively, while $\varphi_1(\xi)$, $\varphi_2(\xi)$ and $\eta(x,t)$ are real functions which represent the shape of the soliton pulse and phase component of the soliton, respectively. Substituting (8) and (9) along with (10) into Eqs. (2) and (3), separating the real and the imaginary parts, we have:

$$\begin{gathered} a_1\varphi_2'' + (k\alpha_1 - \omega)\varphi_1 + \left(\beta_1 - a_1k^2\right)\varphi_2 \\ + \frac{f_1\varphi_1}{b_1\varphi_1^2 + c_1\varphi_2^2} + \frac{d_1\varphi_1}{\sqrt{\varphi_1^2+\varphi_2^2}} + e_1\varphi_1\sqrt{\varphi_1^2+\varphi_2^2} \\ + \left(g_1\varphi_1^2 + h_1\varphi_2^2\right)\varphi_1 = 0, \end{gathered} \tag{11}$$

$$\begin{gathered} a_2\varphi_1'' + (k\alpha_2 - \omega)\varphi_2 + \left(\beta_2 - a_2k^2\right)\varphi_1 \\ + \frac{f_2\varphi_2}{b_2\varphi_2^2 + c_2\varphi_1^2} + \frac{d_2\varphi_2}{\sqrt{\varphi_2^2+\varphi_1^2}} + e_2\varphi_2\sqrt{\varphi_2^2+\varphi_1^2} \\ + \left(g_2\varphi_2^2 + h_2\varphi_1^2\right)\varphi_2 = 0, \end{gathered} \tag{12}$$

and

$$(\alpha_1 - v)\varphi_1' - 2a_1k\varphi_2' = 0, \tag{13}$$

$$(\alpha_2 - v)\varphi_2' - 2a_2k\varphi_1' = 0. \tag{14}$$

Let us set

$$\varphi_2(\xi) = \lambda_1\varphi_1(\xi), \tag{15}$$

where $\lambda_1$ is a non zero constant, such that $\lambda_1 \neq 1$. Consequently, Eqs. (11) – (14) change to:

$$\begin{gathered} a_1\lambda_1\varphi_1\varphi_1'' + \frac{f_1}{\left(b_1 + c_1\lambda_1^2\right)} + \frac{d_1}{\sqrt{1+\lambda_1^2}}\varphi_1 \\ + \left[k\alpha_1 - \omega + \left(\beta_1 - a_1k^2\right)\lambda_1\right]\varphi_1^2 \\ + e_1\sqrt{1+\lambda_1^2}\varphi_1^3 + \left(g_1 + h_1\lambda_1^2\right)\varphi_1^4 = 0, \end{gathered} \tag{16}$$

$$\begin{gathered} a_2\varphi_1\varphi_1'' + \frac{f_2\lambda_1}{\left(b_2\lambda_1^2 + c_2\right)} + \frac{d_2\lambda_1}{\sqrt{1+\lambda_1^2}}\varphi_1 \\ + \left[(k\alpha_2 - \omega)\lambda_1 + \beta_2 - a_2k^2\right]\varphi_1^2 \\ + e_2\lambda_1\sqrt{1+\lambda_1^2}\varphi_1^3 + \left(g_2\lambda_1^2 + h_2\right)\lambda_1\varphi_1^4 = 0, \end{gathered} \tag{17}$$

and

$$\left[(\alpha_1 - v) - 2a_1k\lambda_1\right]\varphi_1' = 0, \tag{18}$$

$$\left[(\alpha_2 - v)\lambda_1 - 2a_2k\right]\varphi_1' = 0. \tag{19}$$

From Eqs. (18) and (19), the velocity of the soliton $v$ is given by:

$$v = \alpha_1 - 2a_1k\lambda_1, \tag{20}$$

and

$$v = \alpha_2 - \frac{2a_2k}{\lambda_1}. \tag{21}$$

From (20) and (21), we have the constraint condition:

$$2a_1k\lambda_1^2 + (\alpha_2 - \alpha_1)\lambda_1 - 2a_2k = 0. \tag{22}$$

Eqs. (16) and (17) have the same form under the constraint conditions:

$$\begin{gathered} a_2 = a_1\lambda_1, \\ f_2\lambda_1\left(b_1 + c_1\lambda_1^2\right) = f_1\left(b_2\lambda_1^2 + c_2\right), \\ d_2\lambda_1 = d_1, \\ (k\alpha_2 - \omega)\lambda_1 + \beta_2 - a_2k^2 = k\alpha_1 - \omega + \left(\beta_1 - a_1k^2\right)\lambda_1, \\ e_2\lambda_1 = e_1, \\ \left(g_2\lambda_1^2 + h_2\right)\lambda_1 = g_1 + h_1\lambda_1^2. \end{gathered} \tag{23}$$

Eq. (16) can be rewritten in the form:

$$\varphi_1\varphi_1'' + \Delta_0 + \Delta_1\varphi_1 + \Delta_2\varphi_1^2 + \Delta_3\varphi_1^3 + \Delta_4\varphi_1^4 = 0, \tag{24}$$

where

$$\begin{gathered} \Delta_0 = \frac{f_1}{a_1\lambda_1\left(b_1 + c_1\lambda_1^2\right)}, \quad \Delta_1 = \frac{d_1}{a_1\lambda_1\sqrt{1+\lambda_1^2}}, \\ \Delta_2 = \frac{k\alpha_1 - \omega + \left(\beta_1 - a_1k^2\right)\lambda_1}{a_1\lambda_1}, \\ \Delta_3 = \frac{e_1\sqrt{1+\lambda_1^2}}{a_1\lambda_1}, \quad \Delta_4 = \frac{\left(g_1 + h_1\lambda_1^2\right)}{a_1\lambda_1}. \end{gathered} \tag{25}$$

In the next subsection, we will solve Eq. (24) using the following method:

### *2.1.1. Extended Kudryashov's Method*

According to this method, we assume that Eq. (24) has the formal solution:

$$\begin{gathered} \varphi_1(\xi) = A_0 + \sum_{s=1}^{N}\sum_{i+j=s} A_{ij}\chi^i(\xi)\psi^j(\xi) \\ + \sum_{s=1}^{N}\sum_{i+j=s} B_{ij}\chi^{-i}(\xi)\psi^{-j}(\xi), \end{gathered} \tag{26}$$

where $A_0, A_{ij}$ and $B_{ij} (i, j = 0,1,2,\ldots,N)$ are constants to be determined, $N$ is a positive integer which determined by balancing the highest order derivatives and the nonlinear terms in Eq. (24), while the functions $\chi(\xi)$ and $\psi(\xi)$ satisfy the Bernoulli and Riccati equations:

$$\chi'(\xi) = R_2\chi^2(\xi) - R_1\chi(\xi), \quad R_2 \neq 0, \tag{27}$$

$$\psi'(\xi) = S_2\psi^2(\xi) + S_1\psi(\xi) + S_0, \quad S_2 \neq 0, \tag{28}$$

respectively, where $R_2, R_1, S_2, S_1$ and $S_0$ are constants. It is well known that the solutions of Eq. (27) are given by:

$$\chi(\xi)=\begin{cases}\dfrac{R_1}{R_2+R_1\exp(R_1\xi+\xi_0)}, & R_1\neq 0,\\ -\dfrac{1}{R_2\xi+\xi_0}, & R_1=0,\end{cases} \tag{29}$$

and the solutions of Eq. (28) are given by:

$$\psi(\xi)=\begin{cases}-\dfrac{1}{2S_2}\left[S_1+\sqrt{\mu}\tanh\left(\dfrac{\sqrt{\mu}}{2}\xi+\xi_0\right)\right], & \text{if } \mu>0\\ -\dfrac{1}{2S_2}\left[S_1+\sqrt{\mu}\coth\left(\dfrac{\sqrt{\mu}}{2}\xi+\xi_0\right)\right], & \text{if } \mu>0\\ -\dfrac{1}{2S_2}\left[S_1-\sqrt{-\mu}\tan\left(\dfrac{\sqrt{-\mu}}{2}\xi+\xi_0\right)\right], & \text{if } \mu<0,\\ -\dfrac{1}{2S_2}\left[S_1+\sqrt{-\mu}\cot\left(\dfrac{\sqrt{-\mu}}{2}\xi+\xi_0\right)\right], & \text{if } \mu<0\\ -\dfrac{S_1}{2S_2}-\dfrac{1}{S_2\xi+\xi_0}, & \text{if } \mu=0\end{cases} \tag{30}$$

where $\mu=S_1^2-4S_0S_2$ and $\xi_0$ is an arbitrary real constant.

Balancing $\varphi_1\varphi_1''$ and $\varphi_1^4$ in Eq. (24), one gets $N=1$. Now, Eq. (24) has the formal solution:

$$\begin{aligned}\varphi_1(\xi)&=A_0+A_{1,0}\chi(\xi)\\&+A_{0,1}\psi(\xi)+B_{1,0}\chi^{-1}(\xi)+B_{0,1}\psi^{-1}(\xi),\end{aligned} \tag{31}$$

where $A_0, A_{1,0}, A_{0,1}, B_{1,0}$ and $B_{0,1}$ are constants to be determined. Substituting (31) along with (27) and (28) into Eq. (24), collecting all the coefficients of $[\psi(\xi)]^l[\chi(\xi)]^m$, $(l,m=0,1,\ldots,8)$ and setting these coefficients equal to zero, one obtains a system of algebraic equations which can be solved using the Maple to get the result:

$$\begin{gathered}A_0=\frac{3S_1\sqrt{-2\Delta_4}-2\Delta_3}{6\Delta_4},\quad A_{0,1}=\frac{S_2}{\Delta_4}\sqrt{-2\Delta_4},\\ A_{1,0}=0,\quad B_{0,1}=0,\quad B_{1,0}=0,\quad R_1=R_1,\\ R_2=R_2,\quad \Delta_0=0,\quad \Delta_1=-\frac{\Delta_3\left(2\Delta_3^2-9\Delta_2\Delta_4\right)}{27\Delta_4^2},\\ S_0=\frac{2\Delta_3^2-6\Delta_2\Delta_4+3S_1^2\Delta_4}{12S_2\Delta_4},\end{gathered} \tag{32}$$

provided

$$\Delta_4<0. \tag{33}$$

Substituting (32) along with (29) and (30) into Eq. (31), one gets the following solutions:

**(I)** Eqs. (2) and (3) have the dark soliton solutions as:

$$u(x,t)=-\frac{1}{3\Delta_4}\left[\Delta_3\pm\sqrt{3\left(\Delta_3^2-3\Delta_2\Delta_4\right)}\tanh\left(\sqrt{-\frac{\Delta_3^2-3\Delta_2\Delta_4}{6\Delta_4}}(x-vt)+\xi_0\right)\right]\exp[i(-kx+\omega t+\theta)], \tag{34}$$

$$v(x,t)=\lambda_1 u(x,t), \tag{35}$$

and the singular soliton solutions as:

$$u(x,t)=-\frac{1}{3\Delta_4}\left[\Delta_3\pm\sqrt{3\left(\Delta_3^2-3\Delta_2\Delta_4\right)}\coth\left(\sqrt{-\frac{\Delta_3^2-3\Delta_2\Delta_4}{6\Delta_4}}(x-vt)+\xi_0\right)\right]\exp[i(-kx+\omega t+\theta)], \tag{36}$$

$$v(x,t)=\lambda_1 u(x,t), \tag{37}$$

provided

$$\Delta_3^2-3\Delta_2\Delta_4>0 \text{ and } \Delta_4<0. \tag{38}$$

The profile of dark solitons (34) and (35) is given by Fig. 1.

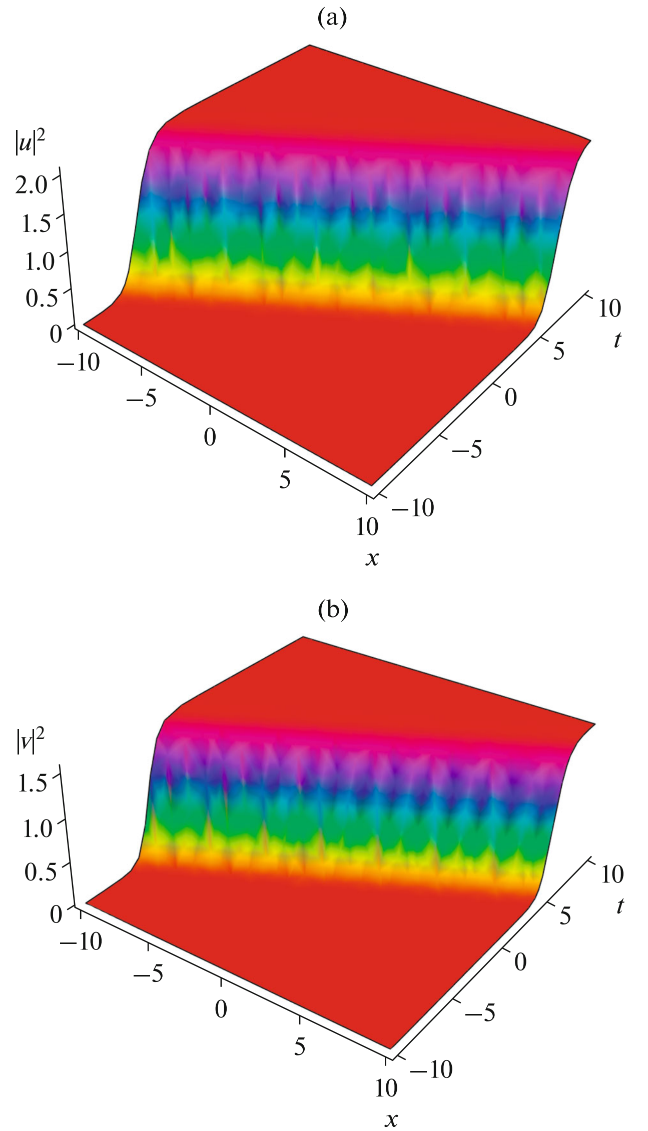



**Fig. 1.** The numerical simulations of the solutions (34) (a) and (35) (b) with the parameter values $a_1 = -0.67$, $a_2 = -0.58$, $e_1 = 0.86$, $g_1 = 0.34$, $h_1 = 0.21$, $k = 0.98$, $v = 1.6$, $\alpha_1 = 0.46$, $\alpha_2 = 0.62$, $\beta_1 = 0.17$, $\lambda_1 = 0.87$, $\xi_0 = 0.13$, $\omega = 0.52$.

**(II)** Eqs. (2) and (3) have the periodic solutions as:

$$u(x,t) = -\frac{1}{3\Delta_4}\left[\Delta_3 \pm \sqrt{-3\left(\Delta_3^2 - 3\Delta_2\Delta_4\right)}\tan\left(\sqrt{\frac{\Delta_3^2 - 3\Delta_2\Delta_4}{6\Delta_4}}(x - vt) + \xi_0\right)\right]\exp[i(-kx + \omega t + \theta)], \tag{39}$$

$$v(x,t) = \lambda_1 u(x,t), \tag{40}$$

and the singular periodic solutions as:

$$u(x,t) = -\frac{1}{3\Delta_4}\left[\Delta_3 \pm \sqrt{-3\left(\Delta_3^2 - 3\Delta_2\Delta_4\right)}\cot\left(\sqrt{\frac{\Delta_3^2 - 3\Delta_2\Delta_4}{6\Delta_4}}(x - vt) + \xi_0\right)\right]\exp[i(-kx + \omega t + \theta)], \tag{41}$$

$$v(x,t) = \lambda_1 u(x,t), \tag{42}$$

provided

$$\Delta_3^2 - 3\Delta_2\Delta_4 < 0 \text{ and } \Delta_4 < 0. \tag{43}$$

**(III)** Eqs. (2) and (3) have the rational solutions as:

$$u(x,t) = \left[-\frac{\Delta_3}{3\Delta_4} - \frac{A_{0,1}\sqrt{-2\Delta_4}}{A_{0,1}\Delta_4(x - vt) + \sqrt{-2\Delta_4}\xi_0}\right] \times \exp[i(-kx + \omega t + \theta_0)], \quad (44)$$

$$v(x,t) = \lambda_1 u(x,t), \quad (45)$$

provided

$$\Delta_0 = 0, \quad \Delta_1 = \frac{\Delta_3^3}{27\Delta_4^2}, \quad \Delta_2 = \frac{\Delta_3^2}{3\Delta_4} \text{ and } \Delta_4 < 0. \quad (46)$$

### *2.2. Chirped Solitons*

To this aim, we introduce the transformation:

$$u(x,t) = \varphi_1(\xi)\exp[i\eta(x,t)], \quad (47)$$

$$v(x,t) = \varphi_2(\xi)\exp[i\eta(x,t)], \quad (48)$$

and

$$\xi = x - vt, \quad \eta(x,t) = \theta(\xi) - \omega t, \quad (49)$$

where $v$ and $\omega$ are all non zero constants to be determined which represent the velocity of soliton and the wave number constants, respectively, while $\theta(\xi)$ represents the phase function. The functions $\varphi_1(\xi)$, $\varphi_2(\xi)$ and $\eta(x,t)$ are real which represent the amplitudes portion of the solitons and the phase component of the soliton, respectively. Substituting (47) and (48) along with (49) into Eqs. (2) and (3), separating the real and the imaginary parts, one gets

$$a_1\varphi_2'' + \omega\varphi_1 + \beta_1\varphi_2 + (v - \alpha_1)\varphi_1\theta' - a_1\varphi_2\theta'^2 + \frac{f_1\varphi_1}{b_1\varphi_1^2 + c_1\varphi_2^2} + \frac{d_1\varphi_1}{\sqrt{\varphi_1^2 + \varphi_2^2}} + e_1\varphi_1\sqrt{\varphi_1^2 + \varphi_2^2} + \left(g_1\varphi_1^2 + h_1\varphi_2^2\right)\varphi_1 = 0, \quad (50)$$

$$a_2\varphi_1'' + \omega\varphi_2 + \beta_2\varphi_1 + (v - \alpha_2)\varphi_2\theta' - a_2\varphi_1\theta'^2 + \frac{f_2\varphi_2}{b_2\varphi_2^2 + c_2\varphi_1^2} + \frac{d_2\varphi_2}{\sqrt{\varphi_2^2 + \varphi_1^2}} + e_2\varphi_2\sqrt{\varphi_2^2 + \varphi_1^2} + \left(g_2\varphi_2^2 + h_2\varphi_1^2\right)\varphi_2 = 0, \quad (51)$$

and

$$(\alpha_1 - v)\varphi_1' + a_1\left[\varphi_2\theta'' + 2\varphi_2'\theta'\right] = 0, \quad (52)$$

$$(\alpha_2 - v)\varphi_2' + a_2\left[\varphi_1\theta'' + 2\varphi_1'\theta'\right] = 0. \quad (53)$$

Let us set

$$\varphi_2(\xi) = \lambda_2\varphi_1(\xi), \quad (54)$$

where $\lambda_2$ is a non zero constant, such that $\lambda_2 \neq 1$. Consequently, Eqs. (50)–(53) reduce to

$$a_1\lambda_2\varphi_1\varphi_1'' + \frac{f_1}{\left(b_1 + c_1\lambda_2^2\right)} + \frac{d_1}{\sqrt{1+\lambda_2^2}}\varphi_1 + \left[\omega + \beta_1\lambda_2 + (v - \alpha_1)\theta' - a_1\lambda_2\theta'^2\right]\varphi_1^2 + e_1\sqrt{1+\lambda_2^2}\varphi_1^3 + \left(g_1 + h_1\lambda_2^2\right)\varphi_1^4 = 0, \quad (55)$$

$$a_2\varphi_1\varphi_1'' + \frac{f_2\lambda_2}{\left(b_2\lambda_2^2 + c_2\right)} + \frac{d_2\lambda_2}{\sqrt{1+\lambda_2^2}}\varphi_1 + \left[\omega\lambda_2 + \beta_2 + (v - \alpha_2)\lambda_2\theta' - a_2\theta'^2\right]\varphi_1^2 + e_2\lambda_2\sqrt{1+\lambda_2^2}\varphi_1^3 + \left(g_2\lambda_2^2 + h_2\right)\lambda_2\varphi_1^4 = 0, \quad (56)$$

and

$$(\alpha_1 - v)\varphi_1' + a_1\lambda_2\left[\varphi_1\theta'' + 2\varphi_1'\theta'\right] = 0, \quad (57)$$

$$(\alpha_2 - v)\lambda_2\varphi_1' + a_2\left[\varphi_1\theta'' + 2\varphi_1'\theta'\right] = 0. \quad (58)$$

By integrating (57) and (58) with zero constants of integration lead to:

$$\theta' = \frac{v - \alpha_1}{2a_1\lambda_2}, \quad (59)$$

or

$$\theta' = \frac{(v - \alpha_2)\lambda_2}{2a_2}. \quad (60)$$

From (59) and (60), one gets the constraint condition:

$$\left(\lambda_2^2 a_1 - a_2\right)v + a_2\alpha_1 - \lambda_2^2 a_1\alpha_2 = 0. \quad (61)$$

From (61), one can obtain the velocity of the soliton $v$ as:

$$v = \frac{\lambda_2^2 a_1\alpha_2 - a_2\alpha_1}{\lambda_2^2 a_1 - a_2}. \quad (62)$$

Then the corresponding chirp is defined by

$$\delta W(x,t) = -\frac{\partial}{\partial x}[\theta(\xi) - \omega t] = -\theta'(\xi), \quad (63)$$

which can be rewritten as

$$\delta W(x,t) = -\left[\frac{v - \alpha_1}{2a_1\lambda_2}\right], \quad (64)$$

where the velocity v is given by (62). Substituting (59) into Eqs. (55) and (56), one gets

$$a_1\lambda_2\varphi_1\varphi_1'' + \frac{f_1}{\left(b_1 + c_1\lambda_2^2\right)} + \frac{d_1}{\sqrt{1+\lambda_2^2}}\varphi_1 + \left[\omega + \beta_1\lambda_2 + \frac{(v - \alpha_1)^2}{4a_1\lambda_2}\right]\varphi_1^2 + e_1\sqrt{1+\lambda_2^2}\varphi_1^3 + \left(g_1 + h_1\lambda_2^2\right)\varphi_1^4 = 0, \quad (65)$$

$$a_2\varphi_1\varphi_1'' + \frac{f_2\lambda_2}{\left(b_2\lambda_2^2 + c_2\right)} + \frac{d_2\lambda_2}{\sqrt{1+\lambda_2^2}}\varphi_1 + \left[\omega\lambda_2 + \beta_2 - \frac{a_2(v-\alpha_1)^2}{4\lambda_2^2 a_1^2} + \frac{(v-\alpha_2)(v-\alpha_1)}{2a_1}\right]\varphi_1^2 + e_2\lambda_2\sqrt{1+\lambda_2^2}\varphi_1^3 + \left(g_2\lambda_2^2 + h_2\right)\lambda_2\varphi_1^4 = 0. \quad (66)$$

Eqs. (65) and (66) have the same form under the constraint conditions:

$$\begin{gathered} a_2 = a_1\lambda_2, \\ f_2\lambda_2\left(b_1 + c_1\lambda_2^2\right) = f_1\left(b_2\lambda_2^2 + c_2\right), \\ d_2\lambda_2 = d_1, \\ 4\lambda_2 a_1(\omega\lambda_2 + \beta_2) - a_2(v-\alpha_1)^2 \\ + 2\lambda_2(v-\alpha_2)(v-\alpha_1) = 4a_1\lambda_2(\omega + \beta_1\lambda_2) + (v-\alpha_1)^2, \\ e_2\lambda_2 = e_1, \\ \left(g_2\lambda_2^2 + h_2\right)\lambda_2 = g_1 + h_1\lambda_2^2. \end{gathered} \quad (67)$$

Eq. (65) can be rewritten in the form:

$$\varphi_1\varphi_1'' + \Gamma_0 + \Gamma_1\varphi_1 + \Gamma_2\varphi_1^2 + \Gamma_3\varphi_1^3 + \Gamma_4\varphi_1^4 = 0, \quad (68)$$

where

$$\Gamma_0 = \frac{f_1}{a_1\lambda_2\left(b_1 + c_1\lambda_2^2\right)}, \quad \Gamma_1 = \frac{d_1}{a_1\lambda_2\sqrt{1+\lambda_2^2}},$$
$$\Gamma_2 = \frac{4a_1\lambda_2(\omega + \beta_1\lambda_2) + (v-\alpha_1)^2}{4a_1^2\lambda_2^2}, \quad (69)$$
$$\Gamma_3 = \frac{e_1\sqrt{1+\lambda_2^2}}{a_1\lambda_2}, \quad \Gamma_4 = \frac{\left(g_1 + h_1\lambda_2^2\right)}{a_1\lambda_2}.$$

In the next subsection, we will solve Eq. (68) using the following method:

### *2.2.1. Extended Kudryashov's Method*

According to this method, one finds that Eq. (68) has the same formal solution (31). Substituting (31) along with (27) and (28) into Eq. (68), collecting all the coefficients of $[\psi(\xi)]^l[\chi(\xi)]^m$, $(l, m = 0, 1, \ldots, 8)$ and setting these coefficients equal to zero, one obtains a system of algebraic equations which can be solved using the Maple to get the result:

$$A_0 = \frac{3S_1\sqrt{-2\Gamma_4} - 2\Gamma_3}{6\Gamma_4}, \quad A_{0,1} = \frac{S_2}{\Gamma_4}\sqrt{-2\Gamma_4},$$
$$A_{1,0} = 0, \quad B_{0,1} = 0, \quad B_{1,0} = 0, \quad R_1 = R_1, \quad R_2 = R_2, \quad \Gamma_0 = 0, \quad \Gamma_1 = -\frac{\Gamma_3\left(2\Gamma_3^2 - 9\Gamma_2\Gamma_4\right)}{27\Gamma_4^2}, \quad (70)$$
$$S_0 = \frac{2\Gamma_3^2 - 6\Gamma_2\Gamma_4 + 3S_1^2\Gamma_4}{12S_2\Gamma_4},$$

provided

$$\Gamma_4 < 0. \quad (71)$$

Substituting (70) along with (29) and (30) into Eq. (31), one gets the following solutions:

**(I)** Eqs. (2) and (3) have the dark soliton solutions as:

$$u(x,t) = -\frac{1}{3\Gamma_4}\left[\Gamma_3 \pm \sqrt{3\left(\Gamma_3^2 - 3\Gamma_2\Gamma_4\right)}\tanh\left(\sqrt{-\frac{\Gamma_3^2 - 3\Gamma_2\Gamma_4}{6\Gamma_4}}(x - vt) + \xi_0\right)\right]\exp\left[i\left(\left(\frac{v-\alpha_1}{2a_1\lambda_2}\right)(x - vt) - \omega t\right)\right], \quad (72)$$
$$v(x,t) = \lambda_2 u(x,t), \quad (73)$$

and the singular soliton solutions as:

$$u(x,t) = -\frac{1}{3\Gamma_4}\left[\Gamma_3 \pm \sqrt{3\left(\Gamma_3^2 - 3\Gamma_2\Gamma_4\right)}\coth\left(\sqrt{-\frac{\Gamma_3^2 - 3\Gamma_2\Gamma_4}{6\Gamma_4}}(x - vt) + \xi_0\right)\right]\exp\left[i\left(\left(\frac{v-\alpha_1}{2a_1\lambda_2}\right)(x - vt) - \omega t\right)\right], \quad (74)$$
$$v(x,t) = \lambda_2 u(x,t), \quad (75)$$

provided

$$\Gamma_3^2 - 3\Gamma_2\Gamma_4 > 0 \text{ and } \Gamma_4 < 0. \quad (76)$$

**(II)** Eqs. (2) and (3) have the periodic solutions as:

$$u(x,t) = -\frac{1}{3\Gamma_4}\left[\Gamma_3 \pm \sqrt{-3\left(\Gamma_3^2 - 3\Gamma_2\Gamma_4\right)}\tan\left(\sqrt{\frac{\Gamma_3^2 - 3\Gamma_2\Gamma_4}{6\Gamma_4}}(x - vt) + \xi_0\right)\right]\exp\left[i\left(\left(\frac{v-\alpha_1}{2a_1\lambda_2}\right)(x - vt) - \omega t\right)\right], \quad (77)$$
$$v(x,t) = \lambda_2 u(x,t), \quad (78)$$

and the singular periodic solutions as:

$$u(x,t) = -\frac{1}{3\Gamma_4}\left[\Gamma_3 \pm \sqrt{-3\left(\Gamma_3^2 - 3\Gamma_2\Gamma_4\right)}\cot\left(\sqrt{\frac{\Gamma_3^2 - 3\Gamma_2\Gamma_4}{6\Gamma_4}}(x - vt) + \xi_0\right)\right]\exp\left[i\left(\left(\frac{v - \alpha_1}{2a_1\lambda_2}\right)(x - vt) - \omega t\right)\right], \quad (79)$$

$$v(x,t) = \lambda_2 u(x,t), \quad (80)$$

provided

$$\Gamma_3^2 - 3\Gamma_2\Gamma_4 < 0 \text{ and } \Gamma_4 < 0. \quad (81)$$

**(III)** Eqs. (2) and (3) have the rational solutions as:

$$u(x,t) = \left[-\frac{\Gamma_3}{3\Gamma_4} - \frac{A_{0,1}\sqrt{-2\Gamma_4}}{A_{0,1}\Gamma_4(x - vt) + \sqrt{-2\Gamma_4}\xi_0}\right]\exp\left[i\left(\left(\frac{v - \alpha_1}{2a_1\lambda_2}\right)(x - vt) - \omega t\right)\right], \quad (82)$$

$$v(x,t) = \lambda_2 u(x,t), \quad (83)$$

provided

$$\Gamma_0 = 0, \quad \Gamma_1 = \frac{\Gamma_3^3}{27\Gamma_4^2}, \quad \Gamma_2 = \frac{\Gamma_3^2}{3\Gamma_4} \text{ and } \Gamma_4 < 0. \quad (84)$$

## 3. MATHEMATICAL ANALYSIS (CASE–II: $n = 2$)

### *3.1. Chirp-Free Solitons*

To this aim, we make the same transformation (8) and (9). Substituting (8) and (9) along with (10) into Eqs. (4) and (5), separating the real and the imaginary parts, we have:

$$\begin{aligned} &a_1\varphi_2'' + (k\alpha_1 - \omega)\varphi_1 + \left(\beta_1 - a_1k^2\right)\varphi_2 \\ &+ \frac{f_1\varphi_1}{b_1\varphi_1^4 + c_1\varphi_1^2\varphi_2^2 + d_1\varphi_2^4} + \frac{e_1\varphi_1}{g_1\varphi_1^2 + h_1\varphi_2^2} \\ &+ \left(r_1\varphi_1^2 + q_1\varphi_2^2\right)\varphi_1 + \left(\xi_1\varphi_1^4 + \zeta_1\varphi_1^2\varphi_2^2 + \eta_1\varphi_2^4\right)\varphi_1 = 0, \end{aligned} \quad (85)$$

$$\begin{aligned} &a_2\varphi_1'' + (k\alpha_2 - \omega)\varphi_2 + \left(\beta_2 - a_2k^2\right)\varphi_1 \\ &+ \frac{f_2\varphi_2}{b_2\varphi_2^4 + c_2\varphi_2^2\varphi_1^2 + d_2\varphi_1^4} + \frac{e_2\varphi_2}{g_2\varphi_2^2 + h_2\varphi_1^2} \\ &+ \left(r_2\varphi_2^2 + q_2\varphi_1^2\right)\varphi_2 + \left(\xi_2\varphi_2^4 + \zeta_2\varphi_2^2\varphi_1^2 + \eta_2\varphi_1^4\right)\varphi_2 = 0, \end{aligned} \quad (86)$$

and

$$\left(\alpha_1 - v\right)\varphi_1' - 2a_1k\varphi_2' = 0, \quad (87)$$

$$\left(\alpha_2 - v\right)\varphi_2' - 2a_2k\varphi_1' = 0. \quad (88)$$

Let us set

$$\varphi_2(\xi) = \lambda_3\varphi_1(\xi), \quad (89)$$

where $\lambda_3$ is a non zero constant, such that $\lambda_3 \neq 1$. Consequently, Eqs. (85)–(88) change to:

$$\begin{aligned} &a_1\lambda_3\varphi_1^3\varphi_1'' + \frac{f_1}{b_1 + c_1\lambda_3^2 + d_1\lambda_3^4} + \frac{e_1}{\left(g_1 + h_1\lambda_3^2\right)}\varphi_1^2 \\ &+ \left[k\alpha_1 - \omega + \left(\beta_1 - a_1k^2\right)\lambda_3\right]\varphi_1^4 \\ &+ \left(r_1 + q_1\lambda_3^2\right)\varphi_1^6 + \left(\xi_1 + \zeta_1\lambda_3^2 + \eta_1\lambda_3^4\right)\varphi_1^8 = 0, \end{aligned} \quad (90)$$

$$\begin{aligned} &a_2\varphi_1^3\varphi_1'' + \frac{f_2\lambda_3}{b_2\lambda_3^4 + c_2\lambda_3^2 + d_2} + \frac{e_2\lambda_3}{\left(g_2\lambda_3^2 + h_2\right)}\varphi_1^2 \\ &+ \left[(k\alpha_2 - \omega)\lambda_3 + \beta_2 - a_2k^2\right]\varphi_1^4 \\ &+ \left(r_2\lambda_3^2 + q_2\right)\lambda_3\varphi_1^6 + \left(\xi_2\lambda_3^4 + \zeta_2\lambda_3^2 + \eta_2\right)\lambda_3\varphi_1^8 = 0, \end{aligned} \quad (91)$$

and

$$\left(\alpha_1 - v\right)\varphi_1' - 2a_1k\lambda_3\varphi_1' = 0, \quad (92)$$

$$\left(\alpha_2 - v\right)\lambda_3\varphi_1' - 2a_2k\varphi_1' = 0. \quad (93)$$

From Eqs. (92) and (93), the velocity of the soliton $v$ is given by:

$$v = \alpha_1 - 2a_1k\lambda_3, \quad (94)$$

and

$$v = \alpha_2 - \frac{2a_2k}{\lambda_3}. \quad (95)$$

From (94) and (95), we have the constraint condition:

$$2a_1k\lambda_3^2 + \left(\alpha_2 - \alpha_1\right)\lambda_3 - 2a_2k = 0. \quad (96)$$

Eqs. (90) and (91) have the same form under the constraint conditions:

$$\begin{gathered} a_2 = a_1\lambda_3, \\ f_2\lambda_3\left(b_1 + c_1\lambda_3^2 + d_1\lambda_3^4\right) = f_1\left(b_2\lambda_3^4 + c_2\lambda_3^2 + d_2\right), \\ e_2\lambda_3\left(g_1 + h_1\lambda_3^2\right) = e_1\left(g_2\lambda_3^2 + h_2\right), \\ (k\alpha_2 - \omega)\lambda_3 + \beta_2 - a_2k^2 = k\alpha_1 - \omega + \left(\beta_1 - a_1k^2\right)\lambda_3, \\ \left(r_2\lambda_3^2 + q_2\right)\lambda_3 = r_1 + q_1\lambda_3^2, \\ \left(\xi_2\lambda_3^4 + \zeta_2\lambda_3^2 + \eta_2\right)\lambda_3 = \xi_1 + \zeta_1\lambda_3^2 + \eta_1\lambda_3^4. \end{gathered} \quad (97)$$

Balancing $\varphi_1^3(\xi)\varphi_1''(\xi)$ with $\varphi_1^8(\xi)$ in Eq. (90) yields $N=\frac{1}{2}$. Since the balance number is not integer, then we take into consideration the transformation:

$$\varphi_1(\xi)=U^{\frac{1}{2}}(\xi), \tag{98}$$

such that $U(\xi)$ is a new function of $\xi$. Substituting (98) into (90), one gets a new equation:

$$\begin{gathered}U'^2-2UU''+\Theta_0+\Theta_1 U\\+\Theta_2U^2+\Theta_3U^3+\Theta_4U^4=0,\end{gathered} \tag{99}$$

where

$$\begin{gathered}\Theta_0=-\frac{4f_1}{a_1\lambda_3\left(b_1+c_1\lambda_3^2+d_1\lambda_3^4\right)},\quad \Theta_1=-\frac{4e_1}{a_1\lambda_3\left(g_1+h_1\lambda_3^2\right)},\\ \Theta_2=-\frac{4\left[k\alpha_1-\omega+\left(\beta_1-a_1k^2\right)\lambda_3\right]}{a_1\lambda_3},\\ \Theta_3=-\frac{4\left(r_1+q_1\lambda_3^2\right)}{a_1\lambda_3},\quad \Theta_4=-\frac{4\left(\xi_1+\zeta_1\lambda_3^2+\eta_1\lambda_3^4\right)}{a_1\lambda_3}.\end{gathered} \tag{100}$$

In the next subsection, we will solve Eq. (99) using the following method.

*3.1.1. Extended Kudryashov's Method*

According to this method, one finds that Eq. (99) has the formal solution

$$\begin{gathered}U(\xi)=A_0+A_{1,0}\chi(\xi)+A_{0,1}\psi(\xi)\\+B_{1,0}\chi^{-1}(\xi)+B_{0,1}\psi^{-1}(\xi).\end{gathered} \tag{101}$$

Substituting (101) along with (27) and (28) into Eq. (99), collecting all the coefficients of $[\psi(\xi)]^l[\chi(\xi)]^m$, $(l,m=0,1,\ldots,8)$ and setting these coefficients equal to zero, one obtains a system of algebraic equations which can be solved using the Maple to get the result:

$$\begin{gathered}A_0=\frac{4S_1\sqrt{3\Theta_4}-3\Theta_3}{8\Theta_4},\quad A_{0,1}=\frac{S_2}{\Theta_4}\sqrt{3\Theta_4},\quad A_{1,0}=0,\\ B_{0,1}=0,\quad B_{1,0}=0,\quad R_1=R_1,\quad R_2=R_2,\\ \Theta_1=0,\quad \Theta_0=-\frac{9\Theta_3^2\left(3\Theta_3^2-32\Theta_2\Theta_4\right)+768\Theta_2^2\Theta_4^2}{1024\Theta_4^3},\\ S_0=\frac{-9\Theta_3^2+32\Theta_2\Theta_4+16S_1^2\Theta_4}{64S_2\Theta_4},\end{gathered} \tag{102}$$

provided $\Theta_4>0$. Substituting (102) along with (29) and (30) into Eq. (101), one gets the following solutions:

**(I)** Eqs. (4) and (5) have the dark soliton solutions as:

$$u(x,t)=\left\{\frac{1}{8\Theta_4}\left[-3\Theta_3+\sqrt{3\left(9\Theta_3^2-32\Theta_2\Theta_4\right)}\tanh\left(\frac{1}{8}\sqrt{\frac{9\Theta_3^2-32\Theta_2\Theta_4}{\Theta_4}}(x-vt)+\xi_0\right)\right]\right\}^{\frac{1}{2}}\exp[i(-kx+\omega t+\theta_0)], \tag{103}$$

$$v(x,t)=\lambda_3u(x,t), \tag{104}$$

and the singular soliton solutions as:

$$u(x,t)=\left\{\frac{1}{8\Theta_4}\left[-3\Theta_3+\sqrt{3\left(9\Theta_3^2-32\Theta_2\Theta_4\right)}\coth\left(\frac{1}{8}\sqrt{\frac{9\Theta_3^2-32\Theta_2\Theta_4}{\Theta_4}}(x-vt)+\xi_0\right)\right]\right\}^{\frac{1}{2}}\exp[i(-kx+\omega t+\theta_0)], \tag{105}$$

$$v(x,t)=\lambda_3u(x,t), \tag{106}$$

provided

$$9\Theta_3^2-32\Theta_2\Theta_4>0 \text{ and } \Theta_4>0. \tag{107}$$

The profile of the solitons (103) and (104) is given by Fig. 2.

**(II)** Eqs. (4) and (5) have the periodic solutions as:

$$u(x,t)=\left\{\frac{1}{8\Theta_4}\left[-3\Theta_3+\sqrt{-3\left(9\Theta_3^2-32\Theta_2\Theta_4\right)}\tan\left(\frac{1}{8}\sqrt{-\frac{9\Theta_3^2-32\Theta_2\Theta_4}{\Theta_4}}(x-vt)+\xi_0\right)\right]\right\}^{\frac{1}{2}}\exp[i(-kx+\omega t+\theta_0)], \tag{108}$$

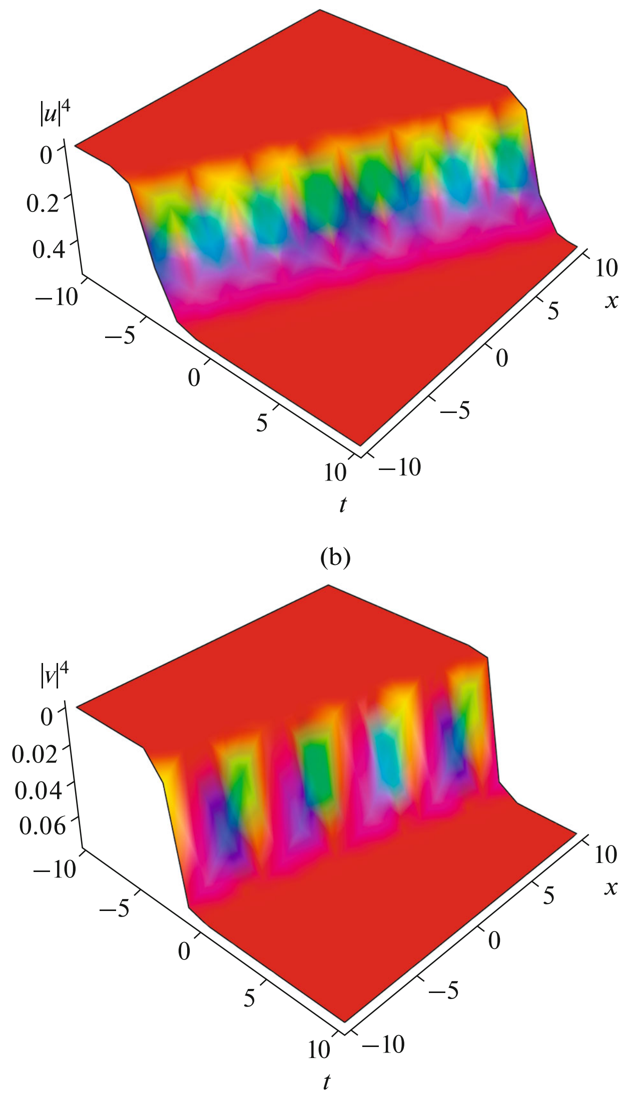
$$v(x,t)=\lambda_3u(x,t), \tag{109}$$

and the singular periodic solutions as:

$$u(x,t)=\left\{\frac{1}{8\Theta_4}\left[-3\Theta_3+\sqrt{-3\left(9\Theta_3^2-32\Theta_2\Theta_4\right)}\cot\left(\frac{1}{8}\sqrt{-\frac{9\Theta_3^2-32\Theta_2\Theta_4}{\Theta_4}}(x-vt)+\xi_0\right)\right]\right\}^{\frac{1}{2}}\exp[i(-kx+\omega t+\theta_0)], \tag{110}$$

$$v(x,t)=\lambda_3u(x,t), \tag{111}$$

(a)




(b)



**Fig. 2.** The numerical simulations of to the solutions (103) (a) and (104) (b) with the parameter values $a_1 = -1.83$, $a_2 = -1.11$, $k = 0.65$, $q_1 = 0.58$, $r_1 = 0.82$, $v = 1.79$, $\alpha_1 = 0.34$, $\alpha_2 = 0.93$, $\beta_1 = 0.4$, $\lambda_3 = 0.61$, $\xi_0 = 0.37$, $\xi_1 = 0.81$, $\zeta_1 = 0.56$, $\eta_1 = 0.69$, $\omega = 0.78$.

provided

$$9\Theta_3^2 - 32\Theta_2\Theta_4 < 0 \text{ and } \Theta_4 > 0. \tag{112}$$

**(III)** Eqs. (4) and (5) have the rational solutions as:

$$u(x,t) = \left[-\frac{3\Theta_3}{8\Theta_4} - \frac{A_{0,1}\sqrt{3\Theta_4}}{A_{0,1}\Theta_4(x-vt)+\sqrt{3\Theta_4}\xi_0}\right]^{\frac{1}{2}} \exp[i(-kx+\omega t+\theta_0)], \tag{113}$$

$$v(x,t) = \lambda_3 u(x,t), \tag{114}$$

provided

$$\Theta_0 = -\frac{27\Theta_3^4}{4096\Theta_4^3}, \quad \Theta_1 = 0, \quad \Theta_2 = \frac{9\Theta_3^2}{32\Theta_4}, \\ \Theta_3 < 0 \text{ and } \Theta_4 > 0. \tag{115}$$

### 3.2. Chirped Solitons

To this aim, we make the same transformation (47) and (48). Substituting (47) and (48) along with (49) into Eqs. (4) and (5), separating the real and the imaginary parts, we have:

$$a_1\varphi_2'' + \omega\varphi_1 + \beta_1\varphi_2 + (v-\alpha_1)\varphi_1\theta' - a_1\varphi_2\theta'^2 + \frac{f_1\varphi_1}{b_1\varphi_1^4 + c_1\varphi_1^2\varphi_2^2 + d_1\varphi_2^4} + \frac{e_1\varphi_1}{g_1\varphi_1^2 + h_1\varphi_2^2} + \left(r_1\varphi_1^2 + q_1\varphi_2^2\right)\varphi_1 + \left(\xi_1\varphi_1^4 + \zeta_1\varphi_1^2\varphi_2^2 + \eta_1\varphi_2^4\right)\varphi_1 = 0, \tag{116}$$

$$a_2\varphi_1'' + \omega\varphi_2 + \beta_2\varphi_1 + (v-\alpha_2)\varphi_2\theta' - a_2\varphi_1\theta'^2 + \frac{f_2\varphi_2}{b_2\varphi_2^4 + c_2\varphi_2^2\varphi_1^2 + d_2\varphi_1^4} + \frac{e_2\varphi_2}{g_2\varphi_2^2 + h_2\varphi_1^2} + \left(r_2\varphi_2^2 + q_2\varphi_1^2\right)\varphi_2 + \left(\xi_2\varphi_2^4 + \zeta_2\varphi_2^2\varphi_1^2 + \eta_2\varphi_1^4\right)\varphi_2 = 0, \tag{117}$$

and

$$(\alpha_1 - v)\varphi_1' + a_1\left[\varphi_2\theta'' + 2\varphi_2'\theta'\right] = 0, \tag{118}$$

$$(\alpha_2 - v)\varphi_2' + a_2\left[\varphi_1\theta'' + 2\varphi_1'\theta'\right] = 0. \tag{119}$$

Let us set

$$\varphi_2(\xi) = \lambda_4\varphi_1(\xi), \tag{120}$$

where $\lambda_4$ is a non zero constant, such that $\lambda_4 \neq 1$. Consequently, Eqs. (116)–(119) change to:

$$a_1\lambda_4\varphi_1^3\varphi_1'' + \left[\frac{f_1}{b_1 + c_1\lambda_4^2 + d_1\lambda_4^4} + \frac{e_1}{\left(g_1 + h_1\lambda_4^2\right)}\right]\varphi_1^2 + \left[\omega + \beta_1\lambda_4 + (v-\alpha_1)\theta' - a_1\lambda_4\theta'^2\right]\varphi_1^4 + \left(r_1 + q_1\lambda_4^2\right)\varphi_1^6 + \left(\xi_1 + \zeta_1\lambda_4^2 + \eta_1\lambda_4^4\right)\varphi_1^8 = 0, \tag{121}$$

$$a_2\varphi_1^3\varphi_1'' + \left[\frac{f_2\lambda_4}{b_2\lambda_4^4 + c_2\lambda_4^2 + d_2} + \frac{e_2\lambda_4}{\left(g_2\lambda_4^2 + h_2\right)}\right]\varphi_1^2 + \left[\omega\lambda_4 + \beta_2 + (v-\alpha_2)\lambda_4\theta' - a_2\theta'^2\right]\varphi_1^4 + \left(r_2\lambda_4^2 + q_2\right)\lambda_4\varphi_1^6 + \left(\xi_2\lambda_4^4 + \zeta_2\lambda_4^2 + \eta_2\right)\lambda_4\varphi_1^8 = 0, \tag{122}$$

and

$$(\alpha_1 - v)\varphi_1' + a_1\lambda_4\left[\varphi_1\theta'' + 2\varphi_1'\theta'\right] = 0, \tag{123}$$

$$(\alpha_2 - v)\lambda_4\varphi_1' + a_2\left[\varphi_1\theta'' + 2\varphi_1'\theta'\right] = 0. \tag{124}$$

By integrating (123) and (124) with zero constants of integration lead to:

$$\theta' = \frac{v-\alpha_1}{2a_1\lambda_4}, \tag{125}$$

or

$$\theta' = \frac{(v-\alpha_2)\lambda_4}{2a_2}. \tag{126}$$

From (125) and (126), one gets the constraint condition:

$$\left(\lambda_4^2 a_1 - a_2\right)v + a_2\alpha_1 - \lambda_4^2 a_1\alpha_2 = 0. \tag{127}$$

From (127), one can obtain the velocity of the soliton $v$ as:

$$v = \frac{\lambda_4^2 a_1\alpha_2 - a_2\alpha_1}{\lambda_4^2 a_1 - a_2}. \tag{128}$$

Then the corresponding chirp is defined by

$$\delta W(x,t) = -\frac{\partial}{\partial x}[\theta(\xi) - \omega t] = -\theta'(\xi), \tag{129}$$

which can be rewritten as

$$\delta W(x,t) = -\left[\frac{v-\alpha_1}{2a_1\lambda_4}\right], \tag{130}$$

where the velocity $v$ is given by (128). Substituting (125) into Eqs. (121) and (122), one gets

$$a_1\lambda_4\varphi_1^3\varphi_1'' + \left[\frac{f_1}{b_1 + c_1\lambda_4^2 + d_1\lambda_4^4} + \frac{e_1}{\left(g_1 + h_1\lambda_4^2\right)}\right]\varphi_1^2 + \left[\omega + \beta_1\lambda_4 + \frac{(v-\alpha_1)^2}{4a_1\lambda_4}\right]\varphi_1^4 + \left(r_1 + q_1\lambda_4^2\right)\varphi_1^6 + \left(\xi_1 + \zeta_1\lambda_4^2 + \eta_1\lambda_4^4\right)\varphi_1^8 = 0, \tag{131}$$

$$a_2\varphi_1^3\varphi_1'' + \left[\frac{f_2\lambda_4}{b_2\lambda_4^4 + c_2\lambda_4^2 + d_2} + \frac{e_2\lambda_4}{\left(g_2\lambda_4^2 + h_2\right)}\right]\varphi_1^2 + \left[\omega\lambda_4 + \beta_2 - \frac{a_2(v-\alpha_1)^2}{4\lambda_4^2 a_1^2} + \frac{(v-\alpha_2)(v-\alpha_1)}{2a_1}\right]\varphi_1^4 + \left(r_2\lambda_4^2 + q_2\right)\lambda_4\varphi_1^6 + \left(\xi_2\lambda_4^4 + \zeta_2\lambda_4^2 + \eta_2\right)\lambda_4\varphi_1^8 = 0. \tag{132}$$

Eqs. (131) and (132) have the same form under the constraint conditions:

$$\begin{gathered} a_2 = a_1\lambda_4, \\ f_2\lambda_4\left(b_1 + c_1\lambda_4^2 + d_1\lambda_4^4\right) = f_1\left(b_2\lambda_4^4 + c_2\lambda_4^2 + d_2\right), \\ e_2\lambda_4\left(g_1 + h_1\lambda_4^2\right) = e_1\left(g_2\lambda_4^2 + h_2\right), \\ 4\lambda_4 a_1(\omega\lambda_4 + \beta_2) - a_2(v-\alpha_1)^2 + 2\lambda_4(v-\alpha_2)(v-\alpha_1) = 4a_1\lambda_4(\omega + \beta_1\lambda_4) + (v-\alpha_1)^2, \\ \left(r_2\lambda_4^2 + q_2\right)\lambda_4 = r_1 + q_1\lambda_4^2, \\ \left(\xi_2\lambda_4^4 + \zeta_2\lambda_4^2 + \eta_2\right)\lambda_4 = \xi_1 + \zeta_1\lambda_4^2 + \eta_1\lambda_4^4. \end{gathered} \tag{133}$$

Balancing $\varphi_1^3(\xi)\varphi_1''(\xi)$ with $\varphi_1^8(\xi)$ in Eq. (131) yields $N = \frac{1}{2}$. Since the balance number is not integer, then we take into consideration the transformation:

$$\varphi_1(\xi) = Z^{\frac{1}{2}}(\xi), \tag{134}$$

such that $Z(\xi)$ is a new function of $\xi$. Substituting (134) into (131), one gets a new equation:

$$Z'^2 - 2ZZ'' + \Pi_0 + \Pi_1 Z + \Pi_2 Z^2 + \Pi_3 Z^3 + \Pi_4 Z^4 = 0, \tag{135}$$

where

$$\Pi_0 = -\frac{4f_1}{a_1\lambda_4\left(b_1 + c_1\lambda_4^2 + d_1\lambda_4^4\right)},$$
$$\Pi_1 = -\frac{4e_1}{a_1\lambda_4\left(g_1 + h_1\lambda_4^2\right)},$$
$$\Pi_2 = -\frac{4a_1\lambda_4(\omega + \beta_1\lambda_4) + (v - \alpha_1)^2}{a_1^2\lambda_4^2}, \tag{136}$$
$$\Pi_3 = -\frac{4\left(r_1 + q_1\lambda_4^2\right)}{a_1\lambda_4}, \quad \Pi_4 = -\frac{4\left(\xi_1 + \zeta_1\lambda_4^2 + \eta_1\lambda_4^4\right)}{a_1\lambda_4}.$$

In the next subsection, we will solve Eq. (135) using the following method:

### *3.2.1. Extended Kudryashov's Method*

According to this method, one finds that Eq. (135) has the formal solution

$$Z(\xi) = A_0 + A_{1,0}\chi(\xi) + A_{0,1}\psi(\xi) + B_{1,0}\chi^{-1}(\xi) + B_{0,1}\psi^{-1}(\xi). \tag{137}$$

Substituting (137) along with (27) and (28) into Eq. (135), collecting all the coefficients of $[\psi(\xi)]^l[\chi(\xi)]^m$, $(l, m = 0, 1, \ldots, 8)$ and setting these coefficients equal to zero, one obtains a system of algebraic equations which can be solved using the Maple to get the result:

$$A_0 = \frac{4S_1\sqrt{3\Pi_4} - 3\Pi_3}{8\Pi_4}, \quad A_{0,1} = \frac{S_2}{\Pi_4}\sqrt{3\Pi_4},$$
$$A_{1,0} = 0, \quad B_{0,1} = 0, \quad B_{1,0} = 0,$$
$$R_1 = R_1, \quad R_2 = R_2, \tag{138}$$
$$\Pi_1 = 0, \quad \Pi_0 = -\frac{9\Pi_3^2\left(3\Pi_3^2 - 32\Pi_2\Pi_4\right) + 768\Pi_2^2\Pi_4^2}{1024\Pi_4^3},$$
$$S_0 = \frac{-9\Pi_3^2 + 32\Pi_2\Pi_4 + 16S_1^2\Pi_4}{64S_2\Pi_4},$$

provided $\Pi_4 > 0$. Substituting (138) along with (29) and (30) into Eq. (137), one gets the following solutions:

**(I)** Eqs. (4) and (5) have the dark soliton solutions as:

$$u(x,t) = \left\{\frac{1}{8\Pi_4}\left[-3\Pi_3 + \sqrt{3\left(9\Pi_3^2 - 32\Pi_2\Pi_4\right)}\tanh\left(\frac{1}{8}\sqrt{\frac{9\Pi_3^2 - 32\Pi_2\Pi_4}{\Pi_4}}(x - vt) + \xi_0\right)\right]\right\}^{\frac{1}{2}} \times \exp\left[i\left(\left(\frac{v - \alpha_1}{2a_1\lambda_4}\right)(x - vt) - \omega t\right)\right], \tag{139}$$

$$v(x,t) = \lambda_4 u(x,t), \tag{140}$$

and the singular soliton solutions as:

$$u(x,t) = \left\{\frac{1}{8\Pi_4}\left[-3\Pi_3 + \sqrt{3\left(9\Pi_3^2 - 32\Pi_2\Pi_4\right)}\coth\left(\frac{1}{8}\sqrt{\frac{9\Pi_3^2 - 32\Pi_2\Pi_4}{\Pi_4}}(x - vt) + \xi_0\right)\right]\right\}^{\frac{1}{2}} \times \exp\left[i\left(\left(\frac{v - \alpha_1}{2a_1\lambda_4}\right)(x - vt) - \omega t\right)\right], \tag{141}$$

$$v(x,t) = \lambda_4 u(x,t), \tag{142}$$

provided

$$9\Pi_3^2 - 32\Pi_2\Pi_4 > 0 \text{ and } \Pi_4 > 0. \tag{143}$$

**(II)** Eqs. (4) and (5) have the periodic solutions as:

$$u(x,t) = \left\{\frac{1}{8\Pi_4}\left[-3\Pi_3 + \sqrt{-3\left(9\Pi_3^2 - 32\Pi_2\Pi_4\right)}\tan\left(\frac{1}{8}\sqrt{-\frac{9\Pi_3^2 - 32\Pi_2\Pi_4}{\Pi_4}}(x - vt) + \xi_0\right)\right]\right\}^{\frac{1}{2}} \times \exp\left[i\left(\left(\frac{v - \alpha_1}{2a_1\lambda_4}\right)(x - vt) - \omega t\right)\right], \tag{144}$$

$$v(x,t) = \lambda_4 u(x,t), \tag{145}$$

and the singular periodic solutions as:

$$u(x,t) = \left\{\frac{1}{8\Pi_4}\left[-3\Pi_3 + \sqrt{-3\left(9\Pi_3^2 - 32\Pi_2\Pi_4\right)}\cot\left(\frac{1}{8}\sqrt{-\frac{9\Pi_3^2 - 32\Pi_2\Pi_4}{\Pi_4}}(x - vt) + \xi_0\right)\right]\right\}^{\frac{1}{2}} \times \exp\left[i\left(\left(\frac{v-\alpha_1}{2a_1\lambda_4}\right)(x - vt) - \omega t\right)\right], \tag{146}$$

$$v(x,t) = \lambda_4 u(x,t), \tag{147}$$

provided

$$9\Pi_3^2 - 32\Pi_2\Pi_4 < 0 \text{ and } \Pi_4 > 0. \tag{148}$$

**(III)** Eqs. (4) and (5) have the rational solutions as:

$$u(x,t) = \left[-\frac{3\Pi_3}{8\Pi_4} - \frac{A_{0,1}\sqrt{3\Pi_4}}{A_{0,1}\Pi_4(x - vt) + \sqrt{3\Pi_4}\xi_0}\right]^{\frac{1}{2}} \exp\left[i\left(\left(\frac{v-\alpha_1}{2a_1\lambda_4}\right)(x - vt) - \omega t\right)\right], \tag{149}$$

$$v(x,t) = \lambda_4 u(x,t), \tag{150}$$

provided

$$\Pi_0 = -\frac{27\Pi_3^4}{4096\Pi_4^3},\quad \Pi_1 = 0, \qquad \Pi_2 = \frac{9\Pi_3^2}{32\Pi_4},\quad \Pi_3 < 0 \text{ and } \Pi_4 > 0. \tag{151}$$

## 4. MATHEMATICAL ANALYSIS (CASE–III: $n = 3$)

### *4.1. Chirp-Free Solitons*

To this aim, we make the same transformation (8) and (9). Substituting (8) and (9) along with (10) into Eqs. (6) and (7), separating the real and the imaginary parts, we have:

$$\begin{aligned} &a_1\varphi_2'' + (k\alpha_1 - \omega)\varphi_1 + \left(\beta_1 - a_1k^2\right)\varphi_2 + \frac{f_1\varphi_1}{b_1\varphi_1^6 + c_1\varphi_1^4\varphi_2^2 + d_1\varphi_1^2\varphi_2^4 + e_1\varphi_2^6} \\ &+ \frac{g_1\varphi_1}{\left(\xi_1\varphi_1^2 + \zeta_1\varphi_2^2\right)\sqrt{\varphi_1^2 + \varphi_2^2}} + \left(l_1\varphi_1^2 + m_1\varphi_2^2\right)\varphi_1 \\ &\times \sqrt{\varphi_1^2 + \varphi_2^2} + \left(q_1\varphi_1^6 + r_1\varphi_1^4\varphi_2^2 + \delta_1\varphi_1^2\varphi_2^4 + \theta_1\varphi_2^6\right)\varphi_1 = 0, \end{aligned} \tag{152}$$

$$\begin{aligned} &a_2\varphi_1'' + (k\alpha_2 - \omega)\varphi_2 + \left(\beta_2 - a_2k^2\right)\varphi_1 + \frac{f_2\varphi_2}{b_2\varphi_2^6 + c_2\varphi_2^4\varphi_1^2 + d_2\varphi_2^2\varphi_1^4 + e_2\varphi_1^6} \\ &+ \frac{g_2\varphi_2}{\left(\xi_2\varphi_2^2 + \zeta_2\varphi_1^2\right)\sqrt{\varphi_2^2 + \varphi_1^2}} + \left(l_2\varphi_2^2 + m_2\varphi_1^2\right)\varphi_2 \\ &\times \sqrt{\varphi_2^2 + \varphi_1^2} + \left(q_2\varphi_2^6 + r_2\varphi_2^4\varphi_1^2 + \delta_2\varphi_2^2\varphi_1^4 + \theta_2\varphi_1^6\right)\varphi_2 = 0, \end{aligned} \tag{153}$$

and

$$(\alpha_1 - v)\varphi_1' - 2a_1k\varphi_2' = 0, \tag{154}$$

$$(\alpha_2 - v)\varphi_2' - 2a_2k\varphi_1' = 0. \tag{155}$$

Let us set

$$\varphi_2(\xi) = \lambda_5\varphi_1(\xi), \tag{156}$$

where $\lambda_5$ is a non zero constant, such that $\lambda_5 \neq 1$. Consequently, Eqs. (152)–(155) change to:

$$\begin{aligned} &a_1\lambda_5\varphi_1^5\varphi_1'' + \frac{f_1}{b_1 + c_1\lambda_5^2 + d_1\lambda_5^4 + e_1\lambda_5^6} + \frac{g_1}{\left(\xi_1 + \zeta_1\lambda_5^2\right)\sqrt{1+\lambda_5^2}}\varphi_1^3 \\ &+ \left[k\alpha_1 - \omega + \left(\beta_1 - a_1k^2\right)\lambda_5\right]\varphi_1^6 + \left(l_1 + m_1\lambda_5^2\right)\sqrt{1+\lambda_5^2}\varphi_1^9 \\ &+ \left(q_1 + r_1\lambda_5^2 + \delta_1\lambda_5^4 + \theta_1\lambda_5^6\right)\varphi_1^{12} = 0, \end{aligned} \tag{157}$$

$$\begin{aligned} &a_2\varphi_1^5\varphi_1'' + \frac{f_2\lambda_5}{b_2\lambda_5^6 + c_2\lambda_5^4 + d_2\lambda_5^2 + e_2} + \frac{g_2\lambda_5}{\left(\xi_2\lambda_5^2 + \zeta_2\right)\sqrt{1+\lambda_5^2}}\varphi_1^3 \\ &+ \left[(k\alpha_2 - \omega)\lambda_5 + \beta_2 - a_2k^2\right]\varphi_1^6 + \left(l_2\lambda_5^2 + m_2\varphi_1^2\right)\lambda_5\sqrt{1+\lambda_5^2}\varphi_1^9 \\ &+ \left(q_2\lambda_5^6 + r_2\lambda_5^4 + \delta_2\lambda_5^2 + \theta_2\right)\lambda_5\varphi_1^{12} = 0, \end{aligned} \tag{158}$$

and

$$(\alpha_1 - v)\varphi_1' - 2a_1k\lambda_5\varphi_1' = 0, \tag{159}$$

$$(\alpha_2 - v)\lambda_5\varphi_1' - 2a_2k\varphi_1' = 0. \tag{160}$$

From Eqs. (159) and (160), the velocity of the soliton $v$ is given by:

$$v = \alpha_1 - 2a_1 k\lambda_5, \tag{161}$$

and

$$v = \alpha_2 - \frac{2a_2 k}{\lambda_5}. \tag{162}$$

From (161) and (162), we have the constraint condition:

$$2a_1 k\lambda_5^2 + (\alpha_2 - \alpha_1)\lambda_5 - 2a_2 k = 0. \tag{163}$$

Eqs. (157) and (158) have the same form under the constraint conditions:

$$\begin{gathered} a_2 = a_1\lambda_5, \\ f_2\lambda_5\left(b_2\lambda_5^6 + c_2\lambda_5^4 + d_2\lambda_5^2 + e_2\right) \\ = f_1\left(b_1 + c_1\lambda_5^2 + d_1\lambda_5^4 + e_1\lambda_5^6\right), \\ g_2\lambda_5\left(\xi_1 + \zeta_1\lambda_5^2\right) = g_1\left(\xi_2\lambda_5^2 + \zeta_2\right), \\ (k\alpha_2 - \omega)\lambda_5 + \beta_2 - a_2 k^2 \\ = k\alpha_1 - \omega + \left(\beta_1 - a_1 k^2\right)\lambda_5, \\ \left(l_2\lambda_5^2 + m_2\varphi_1^2\right)\lambda_5 = l_1 + m_1\lambda_5^2, \\ q_1 + r_1\lambda_5^2 + \delta_1\lambda_5^4 + \theta_1\lambda_5^6 = q_1 + r_1\lambda_5^2 + \delta_1\lambda_5^4 + \theta_1\lambda_5^6. \end{gathered} \tag{164}$$

Balancing $\varphi_1^5(\xi)\varphi_1''(\xi)$ with $\varphi_1^{12}(\xi)$ in Eq. (157) yields $N = \frac{1}{3}$. Since the balance number is not integer, then we take into consideration the transformation:

$$\varphi_1(\xi) = H^{\frac{1}{3}}(\xi), \tag{165}$$

such that $H(\xi)$ is a new function of $\xi$. Substituting (165) into (157), one gets a new equation:

$$\begin{gathered} 3HH'' - 2H'^2 + \Omega_0 + \Omega_1 H \\ + \Omega_2 H^2 + \Omega_3 H^3 + \Omega_4 H^4 = 0, \end{gathered} \tag{166}$$

where

$$\begin{gathered} \Omega_0 = \frac{9f_1}{a_1\lambda_5\left(b_1 + c_1\lambda_5^2 + d_1\lambda_5^4 + e_1\lambda_5^6\right)}, \\ \Omega_1 = \frac{9g_1}{a_1\lambda_5\left(\xi_1 + \zeta_1\lambda_5^2\right)\sqrt{1 + \lambda_5^2}}, \\ \Omega_2 = \frac{9\left[k\alpha_1 - \omega + \left(\beta_1 - a_1 k^2\right)\lambda_5\right]}{a_1\lambda_5}, \\ \Omega_3 = \frac{9\left(l_1 + m_1\lambda_5^2\right)\sqrt{1 + \lambda_5^2}}{a_1\lambda_5}, \\ \Omega_4 = \frac{9\left(q_1 + r_1\lambda_5^2 + \delta_1\lambda_5^4 + \theta_1\lambda_5^6\right)}{a_1\lambda_5}. \end{gathered} \tag{167}$$

In the next subsection, we will solve Eq. (166) using the following method.

#### *4.1.1. Extended Kudryashov's Method*

According to this method, one finds that Eq. (166) has the formal solution

$$\begin{gathered} H(\xi) = A_0 + A_{1,0}\chi(\xi) \\ + A_{0,1}\psi(\xi) + B_{1,0}\chi^{-1}(\xi) + B_{0,1}\psi^{-1}(\xi). \end{gathered} \tag{168}$$

Substituting (168) along with (27) and (28) into Eq. (166), collecting all the coefficients of $[\psi(\xi)]^l[\chi(\xi)]^m$, $(l, m = 0, 1, \ldots, 8)$ and setting these coefficients equal to zero, one obtains a system of algebraic equations which can be solved using the Maple to get the result:

$$\begin{gathered} A_0 = A_0,\quad A_{0,1} = 0,\quad A_{1,0} = 0,\quad B_{0,1} = -\frac{2S_0}{\sqrt{-\Omega_4}},\quad B_{1,0} = 0,\quad R_1 = R_1,\quad R_2 = R_2,\quad S_0 = S_0, \\ S_1 = \frac{5A_0\Omega_4 + 2\Omega_3}{5\sqrt{-\Omega_4}},\quad S_2 = \frac{5\Omega_4\left(5\Omega_1 - \Omega_3 A_0^2\right) - 4A_0\Omega_3^2}{20S_0\Omega_3}, \\ \Omega_0 = -\frac{25\Omega_4\Omega_1^2}{2\Omega_3^2},\quad \Omega_0 = \frac{8\Omega_3^3 - 125\Omega_1\Omega_4^2}{50\Omega_3\Omega_4}, \end{gathered} \tag{169}$$

provided $\Omega_4 < 0$, $\Omega_3 \neq 0$ and $S_0 \neq 0$. Substituting (169) along with (29) and (30) into Eq. (168), one gets the following solutions:

**(I)** Eqs. (6) and (7) have the dark soliton solutions as:

$$u(x,t) = \left[\frac{A_0\sqrt{\Omega_3\left(4\Omega_3^3 + 125\Omega_1\Omega_4^2\right)}\tanh\left(\frac{1}{10}\sqrt{-\frac{4\Omega_3^3 + 125\Omega_1\Omega_4^2}{\Omega_3\Omega_4}}(x - vt) + \xi_0\right) - 25\Omega_1\Omega_4 + 2A_0\Omega_3^2}{\sqrt{\Omega_3\left(4\Omega_3^3 + 125\Omega_1\Omega_4^2\right)}\tanh\left(\frac{1}{10}\sqrt{-\frac{4\Omega_3^3 + 125\Omega_1\Omega_4^2}{\Omega_3\Omega_4}}(x - vt) + \xi_0\right) - 2\Omega_3^2 - 5A_0\Omega_3\Omega_4}\right]^{\frac{1}{3}} \times \exp[i(-kx + \omega t + \theta_0)], \tag{170}$$

$$v(x,t) = \lambda_5 u(x,t), \tag{171}$$

and the singular soliton solutions as:

$$u(x,t)=\left[\frac{A_0\sqrt{\Omega_3\left(4\Omega_3^3+125\Omega_1\Omega_4^2\right)}\coth\left(\frac{1}{10}\sqrt{-\frac{4\Omega_3^3+125\Omega_1\Omega_4^2}{\Omega_3\Omega_4}}(x-vt)+\xi_0\right)-25\Omega_1\Omega_4+2A_0\Omega_3^2}{\sqrt{\Omega_3\left(4\Omega_3^3+125\Omega_1\Omega_4^2\right)}\coth\left(\frac{1}{10}\sqrt{-\frac{4\Omega_3^3+125\Omega_1\Omega_4^2}{\Omega_3\Omega_4}}(x-vt)+\xi_0\right)-2\Omega_3^2-5A_0\Omega_3\Omega_4}\right]^{\frac{1}{3}} \times\exp[i(-kx+\omega t+\theta_0)], \tag{172}$$

$$v(x,t)=\lambda_5u(x,t), \tag{173}$$

provided

$$\Omega_3\left(4\Omega_3^3+125\Omega_1\Omega_4^2\right)>0 \text{ and } \Omega_4<0. \tag{174}$$

The profile of the solitons (170) and (171) is given by Fig. 3.

**(II)** Eqs. (6) and (7) have the periodic solutions as:

$$u(x,t)=\left[\frac{A_0\sqrt{-\Omega_3\left(4\Omega_3^3+125\Omega_1\Omega_4^2\right)}\tan\left(\frac{1}{10}\sqrt{\frac{4\Omega_3^3+125\Omega_1\Omega_4^2}{\Omega_3\Omega_4}}(x-vt)+\xi_0\right)-25\Omega_1\Omega_4+2A_0\Omega_3^2}{\sqrt{-\Omega_3\left(4\Omega_3^3+125\Omega_1\Omega_4^2\right)}\tan\left(\frac{1}{10}\sqrt{\frac{4\Omega_3^3+125\Omega_1\Omega_4^2}{\Omega_3\Omega_4}}(x-vt)+\xi_0\right)-2\Omega_3^2-5A_0\Omega_3\Omega_4}\right]^{\frac{1}{3}} \times\exp[i(-kx+\omega t+\theta_0)], \tag{175}$$

$$v(x,t)=\lambda_5u(x,t), \tag{176}$$

and the singular periodic solutions as:

$$u(x,t)=\left[\frac{A_0\sqrt{-\Omega_3\left(4\Omega_3^3+125\Omega_1\Omega_4^2\right)}\cot\left(\frac{1}{10}\sqrt{\frac{4\Omega_3^3+125\Omega_1\Omega_4^2}{\Omega_3\Omega_4}}(x-vt)+\xi_0\right)-25\Omega_1\Omega_4+2A_0\Omega_3^2}{\sqrt{-\Omega_3\left(4\Omega_3^3+125\Omega_1\Omega_4^2\right)}\cot\left(\frac{1}{10}\sqrt{\frac{4\Omega_3^3+125\Omega_1\Omega_4^2}{\Omega_3\Omega_4}}(x-vt)+\xi_0\right)-2\Omega_3^2-5A_0\Omega_3\Omega_4}\right]^{\frac{1}{3}} \times\exp[i(-kx+\omega t+\theta_0)], \tag{177}$$

$$v(x,t)=\lambda_5u(x,t), \tag{178}$$

provided

$$\Omega_3\left(4\Omega_3^3+125\Omega_1\Omega_4^2\right)<0 \text{ and } \Omega_4<0. \tag{179}$$

**(III)** Eqs. (6) and (7) have the rational solutions as:

$$u(x,t)=\left[A_0-\frac{2\Omega_3+5A_0\Omega_4}{5\Omega_4}\left\{1+\frac{10\Omega_4}{(2\Omega_3+5A_0\Omega_4)\sqrt{-\Omega_4}\,(x-vt)-10\Omega_4\left(5\Omega_4B_{0,1}\xi_0-1\right)}\right\}\right]^{\frac{1}{3}} \times\exp[i(-kx+\omega t+\theta_0)], \tag{180}$$

$$v(x,t)=\lambda_5u(x,t), \tag{181}$$

provided

$$\Omega_0=-\frac{8\Omega_3^4}{625\Omega_4^3},\quad \Omega_1=-\frac{4\Omega_3^3}{125\Omega_4^2}, \quad \Omega_2=\frac{6\Omega_3^2}{25\Omega_4} \text{ and } \Omega_4<0. \tag{182}$$

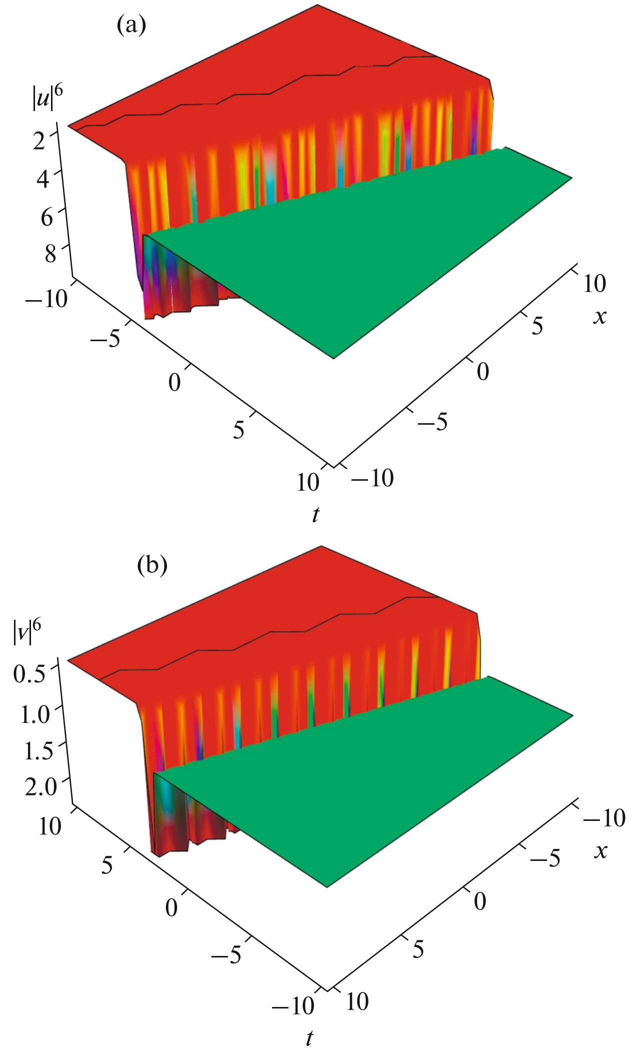



**Fig. 3.** The numerical simulations of the solutions (170) (a) and (171) (b) with the parameter values $a_1 = -2.47$, $a_2 = -1.95$, $g_1 = 0.74$, $k = 0.46$, $l_1 = 0.35$, $m_1 = 0.98$, $q_1 = 0.87$, $r_1 = 0.15$, $v = 2.2$, $\alpha_1 = 0.41$, $\alpha_2 = 0.69$, $\delta_1 = 0.26$, $\lambda_5 = 0.79$, $\xi_0 = 0.52$, $\xi_1 = 0.63$, $\zeta_1 = 0.29$, $\theta_1 = 0.14$, $A_0 = 0.59$.

### *4.2. Chirped Solitons*

To this aim, we make the same transformation (47) and (48). Substituting (47) and (48) along with (49) into Eqs. (6) and (7), separating the real and the imaginary parts, we have:

$$\begin{aligned}&a_1\varphi_2'' + \omega\varphi_1 + \beta_1\varphi_2 + (v-\alpha_1)\varphi_1\theta' \\ &- a_1\varphi_2\theta'^2 + \frac{f_1\varphi_1}{b_1\varphi_1^6 + c_1\varphi_1^4\varphi_2^2 + d_1\varphi_1^2\varphi_2^4 + e_1\varphi_2^6} \\ &+ \frac{g_1\varphi_1}{\left(\xi_1\varphi_1^2 + \zeta_1\varphi_2^2\right)\sqrt{\varphi_1^2+\varphi_2^2}} + \left(l_1\varphi_1^2 + m_1\varphi_2^2\right)\varphi_1 \\ &\times\sqrt{\varphi_1^2+\varphi_2^2} + \left(q_1\varphi_1^6 + r_1\varphi_1^4\varphi_2^2 + \delta_1\varphi_1^2\varphi_2^4 + \theta_1\varphi_2^6\right)\varphi_1 = 0,\end{aligned} \tag{183}$$

$$\begin{aligned}&a_2\varphi_1'' + \omega\varphi_2 + \beta_2\varphi_1 + (v-\alpha_2)\varphi_2\theta' \\ &- a_2\varphi_1\theta'^2 + \frac{f_2\varphi_2}{b_2\varphi_2^6 + c_2\varphi_2^4\varphi_1^2 + d_2\varphi_2^2\varphi_1^4 + e_2\varphi_1^6} \\ &+ \frac{g_2\varphi_2}{\left(\xi_2\varphi_2^2 + \zeta_2\varphi_1^2\right)\sqrt{\varphi_2^2+\varphi_1^2}} \\ &+ \left(l_2\varphi_2^2 + m_2\varphi_1^2\right)\varphi_2\sqrt{\varphi_2^2+\varphi_1^2} \\ &+ \left(q_2\varphi_2^6 + r_2\varphi_2^4\varphi_1^2 + \delta_2\varphi_2^2\varphi_1^4 + \theta_2\varphi_1^6\right)\varphi_2 = 0,\end{aligned} \tag{184}$$

and

$$(\alpha_1 - v)\varphi_1' + a_1\left[\varphi_2\theta'' + 2\varphi_2'\theta'\right] = 0, \tag{185}$$

$$(\alpha_2 - v)\varphi_2' + a_2\left[\varphi_1\theta'' + 2\varphi_1'\theta'\right] = 0. \tag{186}$$

Let us set

$$\varphi_2(\xi) = \lambda_6\varphi_1(\xi), \tag{187}$$

where $\lambda_6$ is a non zero constant, such that $\lambda_6 \neq 1$. Consequently, Eqs. (183)–(186) change to:

$$\begin{aligned}&a_1\lambda_6\varphi_1^5\varphi_1'' + \frac{f_1}{b_1 + c_1\lambda_6^2 + d_1\lambda_6^4 + e_1\lambda_6^6} \\ &+ \frac{g_1}{\left(\xi_1 + \zeta_1\lambda_6^2\right)\sqrt{1+\lambda_6^2}}\varphi_1^3 \\ &+ \left[\omega + \beta_1\lambda_6 + (v-\alpha_1)\theta' - a_1\lambda_6\theta'^2\right]\varphi_1^6 \\ &+ \left(l_1 + m_1\lambda_6^2\right)\sqrt{1+\lambda_6^2}\varphi_1^9 \\ &+ \left(q_1 + r_1\lambda_6^2 + \delta_1\lambda_6^4 + \theta_1\lambda_6^6\right)\varphi_1^{12} = 0,\end{aligned} \tag{188}$$

$$\begin{aligned}&a_2\varphi_1^5\varphi_1'' + \frac{f_2\lambda_6}{b_2\lambda_6^6 + c_2\lambda_6^4 + d_2\lambda_6^2 + e_2} \\ &+ \frac{g_2\lambda_6}{\left(\xi_2\lambda_6^2 + \zeta_2\right)\sqrt{1+\lambda_6^2}}\varphi_1^3 \\ &+ \left[\omega\lambda_6 + \beta_2 + (v-\alpha_2)\lambda_6\theta' - a_2\theta'^2\right]\varphi_1^6 \\ &+ \left(l_2\lambda_6^2 + m_2\varphi_1^2\right)\lambda_6\sqrt{1+\lambda_6^2}\varphi_1^9 \\ &+ \left(q_2\lambda_6^6 + r_2\lambda_6^4 + \delta_2\lambda_6^2 + \theta_2\right)\lambda_6\varphi_1^{12} = 0,\end{aligned} \tag{189}$$

and

$$(\alpha_1 - v)\varphi_1' + a_1\lambda_6\left[\varphi_1\theta'' + 2\varphi_1'\theta'\right] = 0, \tag{190}$$

$$(\alpha_2 - v)\lambda_6\varphi_1' + a_2\left[\varphi_1\theta'' + 2\varphi_1'\theta'\right] = 0. \tag{191}$$

By integrating (190) and (191) with zero constants of integration lead to:

$$\theta' = \frac{v-\alpha_1}{2a_1\lambda_6}, \tag{192}$$

or

$$\theta' = \frac{(v-\alpha_2)\lambda_6}{2a_2}. \tag{193}$$

From (192) and (193), one gets the constraint condition:

$$\left(\lambda_6^2 a_1 - a_2\right)v + a_2\alpha_1 - \lambda_6^2 a_1\alpha_2 = 0. \tag{194}$$

From (194), one can obtain the velocity of the soliton $v$ as:

$$v = \frac{\lambda_6^2 a_1\alpha_2 - a_2\alpha_1}{\lambda_6^2 a_1 - a_2}. \tag{195}$$

Then the corresponding chirp is defined by

$$\delta W(x,t) = -\frac{\partial}{\partial x}[\theta(\xi) - \omega t] = -\theta'(\xi), \tag{196}$$

which can be rewritten as

$$\delta W(x,t) = -\left[\frac{v-\alpha_1}{2a_1\lambda_6}\right], \tag{197}$$

where the velocity $v$ is given by (195). Substituting (192) into Eqs. (188) and (189), one gets

$$\begin{aligned} & a_1\lambda_6\varphi_1^5\varphi_1'' + \frac{f_1}{b_1 + c_1\lambda_6^2 + d_1\lambda_6^4 + e_1\lambda_6^6} \\ & + \frac{g_1}{\left(\xi_1 + \zeta_1\lambda_6^2\right)\sqrt{1+\lambda_6^2}}\varphi_1^3 + \left[\omega + \beta_1\lambda_6 + \frac{(v-\alpha_1)^2}{4a_1\lambda_6}\right] \\ & \times \varphi_1^6 + \left(l_1 + m_1\lambda_6^2\right)\sqrt{1+\lambda_6^2}\varphi_1^9 \\ & + \left(q_1 + r_1\lambda_6^2 + \delta_1\lambda_6^4 + \theta_1\lambda_6^6\right)\varphi_1^{12} = 0, \end{aligned} \tag{198}$$

$$\begin{aligned} & a_2\varphi_1^5\varphi_1'' + \frac{f_2\lambda_6}{b_2\lambda_6^6 + c_2\lambda_6^4 + d_2\lambda_6^2 + e_2} \\ & + \frac{g_2\lambda_6}{\left(\xi_2\lambda_6^2 + \zeta_2\right)\sqrt{1+\lambda_6^2}}\varphi_1^3 \\ & + \left[\omega\lambda_6 + \beta_2 - \frac{a_2(v-\alpha_1)^2}{4\lambda_6^2 a_1^2} + \frac{(v-\alpha_2)(v-\alpha_1)}{2a_1}\right]\varphi_1^6 \\ & + \left(l_2\lambda_6^2 + m_2\varphi_1^2\right)\lambda_6\sqrt{1+\lambda_6^2}\varphi_1^9 \\ & + \left(q_2\lambda_6^6 + r_2\lambda_6^4 + \delta_2\lambda_6^2 + \theta_2\right)\lambda_6\varphi_1^{12} = 0. \end{aligned} \tag{199}$$

Eqs. (198) and (199) have the same form under the constraint conditions:

$$\begin{aligned} & a_2 = a_1\lambda_6, \\ & f_2\lambda_6\left(b_1 + c_1\lambda_6^2 + d_1\lambda_6^4 + e_1\lambda_6^6\right) \\ & = f_1\left(b_2\lambda_6^6 + c_2\lambda_6^4 + d_2\lambda_6^2 + e_2\right), \\ & g_2\lambda_6\left(\xi_1 + \zeta_1\lambda_6^2\right) = g_1\left(\xi_2\lambda_6^2 + \zeta_2\right), \\ & 4\lambda_6 a_1(\omega\lambda_6 + \beta_2) - a_2(v-\alpha_1)^2 + 2\lambda_6(v-\alpha_2) \\ & \times(v-\alpha_1) = 4a_1\lambda_6(\omega + \beta_1\lambda_6) + (v-\alpha_1)^2, \\ & \left(l_2\lambda_6^2 + m_2\varphi_1^2\right)\lambda_6 = l_1 + m_1\lambda_6^2, \\ & \left(q_2\lambda_6^6 + r_2\lambda_6^4 + \delta_2\lambda_6^2 + \theta_2\right)\lambda_6 \\ & = q_1 + r_1\lambda_6^2 + \delta_1\lambda_6^4 + \theta_1\lambda_6^6. \end{aligned} \tag{200}$$

Balancing $\varphi_1^5(\xi)\varphi_1''(\xi)$ with $\varphi_1^{12}(\xi)$ in Eq. (198) yields $N = \frac{1}{3}$. Since the balance number is not integer, then we take into consideration the transformation:

$$\varphi_1(\xi) = Q^{\frac{1}{3}}(\xi), \tag{201}$$

such that $Q(\xi)$ is a new function of $\xi$. Substituting (201) into (198), one gets a new equation:

$$\begin{aligned} & 3QQ'' - 2Q'^2 + \Upsilon_0 \\ & + \Upsilon_1 Q + \Upsilon_2 Q^2 + \Upsilon_3 Q^3 + \Upsilon_4 Q^4 = 0, \end{aligned} \tag{202}$$

where

$$\begin{aligned} & \Upsilon_0 = \frac{9f_1}{a_1\lambda_6\left(b_1 + c_1\lambda_6^2 + d_1\lambda_6^4 + e_1\lambda_6^6\right)}, \\ & \Upsilon_1 = \frac{9g_1}{a_1\lambda_6\left(\xi_1 + \zeta_1\lambda_6^2\right)\sqrt{1+\lambda_6^2}}, \\ & \Upsilon_2 = \frac{9a_1\lambda_6(\omega + \beta_1\lambda_6) + (v-\alpha_1)^2}{a_1^2\lambda_6^2}, \\ & \Upsilon_3 = \frac{9\left(l_1 + m_1\lambda_6^2\right)\sqrt{1+\lambda_6^2}}{a_1\lambda_6}, \\ & \Upsilon_4 = \frac{9\left(q_1 + r_1\lambda_6^2 + \delta_1\lambda_6^4 + \theta_1\lambda_6^6\right)}{a_1\lambda_6}. \end{aligned} \tag{203}$$

In the next subsection, we will solve Eq. (202) using the following method:

#### *4.2.1. Extended Kudryashov's Method*

According to this method, one finds that Eq. (202) has the formal solution

$$\begin{aligned} & Q(\xi) = A_0 + A_{1,0}\chi(\xi) \\ & + A_{0,1}\psi(\xi) + B_{1,0}\chi^{-1}(\xi) + B_{0,1}\psi^{-1}(\xi). \end{aligned} \tag{204}$$

Substituting (204) along with (27) and (28) into Eq. (202), collecting all the coefficients of $[\psi(\xi)]^l[\chi(\xi)]^m$, $(l, m = 0, 1, \ldots, 8)$ and setting these coefficients equal to zero, one obtains a system of algebraic equations which can be solved using the Maple to get the result:

$$\begin{aligned} & A_0 = A_0,\quad A_{0,1} = 0,\quad A_{1,0} = 0,\quad B_{0,1} = -\frac{2S_0}{\sqrt{-\Upsilon_4}}, \\ & B_{1,0} = 0,\quad R_1 = R_1,\quad R_2 = R_2,\quad S_0 = S_0, \\ & S_1 = \frac{5A_0\Upsilon_4 + 2\Upsilon_3}{5\sqrt{-\Upsilon_4}}, \\ & S_2 = \frac{5\Upsilon_4\left(5\Upsilon_1 - \Upsilon_3 A_0^2\right) - 4A_0\Upsilon_3^2}{20S_0\Upsilon_3}, \\ & \Upsilon_0 = -\frac{25\Upsilon_4\Upsilon_1^2}{2\Upsilon_3^2},\quad \Upsilon_0 = \frac{8\Upsilon_3^3 - 125\Upsilon_1\Upsilon_4^2}{50\Upsilon_3\Upsilon_4}, \end{aligned} \tag{205}$$

provided $\Upsilon_4 < 0$, $\Upsilon_3 \neq 0$ and $S_0 \neq 0$. Substituting (205) along with (29) and (30) into Eq. (204), one gets the following solutions:

**(I)** Eqs. (6) and (7) have the dark soliton solutions as:

$$u(x,t) = \left[\frac{A_0\sqrt{\Upsilon_3\left(4\Upsilon_3^3+125\Upsilon_1\Upsilon_4^2\right)}\tanh\left(\frac{1}{10}\sqrt{-\frac{4\Upsilon_3^3+125\Upsilon_1\Upsilon_4^2}{\Upsilon_3\Upsilon_4}}(x-vt)+\xi_0\right)-25\Upsilon_1\Upsilon_4+2A_0\Upsilon_3^2}{\sqrt{\Upsilon_3\left(4\Upsilon_3^3+125\Upsilon_1\Upsilon_4^2\right)}\tanh\left(\frac{1}{10}\sqrt{-\frac{4\Upsilon_3^3+125\Upsilon_1\Upsilon_4^2}{\Upsilon_3\Upsilon_4}}(x-vt)+\xi_0\right)-2\Upsilon_3^2-5A_0\Upsilon_3\Upsilon_4}\right]^{\frac{1}{3}} \times \exp\left[i\left(\left(\frac{v-\alpha_1}{2a_1\lambda_6}\right)(x-vt)-\omega t\right)\right], \tag{206}$$

$$v(x,t) = \lambda_6 u(x,t) \tag{207}$$

and the singular soliton solutions as:

$$u(x,t) = \left[\frac{A_0\sqrt{\Upsilon_3\left(4\Upsilon_3^3+125\Upsilon_1\Upsilon_4^2\right)}\coth\left(\frac{1}{10}\sqrt{-\frac{4\Upsilon_3^3+125\Upsilon_1\Upsilon_4^2}{\Upsilon_3\Upsilon_4}}(x-vt)+\xi_0\right)-25\Upsilon_1\Upsilon_4+2A_0\Upsilon_3^2}{\sqrt{\Upsilon_3\left(4\Upsilon_3^3+125\Upsilon_1\Upsilon_4^2\right)}\coth\left(\frac{1}{10}\sqrt{-\frac{4\Upsilon_3^3+125\Upsilon_1\Upsilon_4^2}{\Upsilon_3\Upsilon_4}}(x-vt)+\xi_0\right)-2\Upsilon_3^2-5A_0\Upsilon_3\Upsilon_4}\right]^{\frac{1}{3}} \times \exp\left[i\left(\left(\frac{v-\alpha_1}{2a_1\lambda_6}\right)(x-vt)-\omega t\right)\right], \tag{208}$$

$$v(x,t) = \lambda_6 u(x,t), \tag{209}$$

provided

$$\Upsilon_3\left(4\Upsilon_3^3+125\Upsilon_1\Upsilon_4^2\right) > 0 \text{ and } \Upsilon_4 < 0. \tag{210}$$

**(II)** Eqs. (6) and (7) have the periodic solutions as:

$$u(x,t) = \left[\frac{A_0\sqrt{-\Upsilon_3\left(4\Upsilon_3^3+125\Upsilon_1\Upsilon_4^2\right)}\tan\left(\frac{1}{10}\sqrt{\frac{4\Upsilon_3^3+125\Upsilon_1\Upsilon_4^2}{\Upsilon_3\Upsilon_4}}(x-vt)+\xi_0\right)-25\Upsilon_1\Upsilon_4+2A_0\Upsilon_3^2}{\sqrt{-\Upsilon_3\left(4\Upsilon_3^3+125\Upsilon_1\Upsilon_4^2\right)}\tan\left(\frac{1}{10}\sqrt{\frac{4\Upsilon_3^3+125\Upsilon_1\Upsilon_4^2}{\Upsilon_3\Upsilon_4}}(x-vt)+\xi_0\right)-2\Upsilon_3^2-5A_0\Upsilon_3\Upsilon_4}\right]^{\frac{1}{3}} \times \exp\left[i\left(\left(\frac{v-\alpha_1}{2a_1\lambda_6}\right)(x-vt)-\omega t\right)\right], \tag{211}$$

$$v(x,t) = \lambda_6 u(x,t), \tag{212}$$

and the singular periodic solutions as:

$$u(x,t) = \left[\frac{A_0\sqrt{-\Upsilon_3\left(4\Upsilon_3^3+125\Upsilon_1\Upsilon_4^2\right)}\cot\left(\frac{1}{10}\sqrt{\frac{4\Upsilon_3^3+125\Upsilon_1\Upsilon_4^2}{\Upsilon_3\Upsilon_4}}(x-vt)+\xi_0\right)-25\Upsilon_1\Upsilon_4+2A_0\Upsilon_3^2}{\sqrt{-\Upsilon_3\left(4\Upsilon_3^3+125\Upsilon_1\Upsilon_4^2\right)}\cot\left(\frac{1}{10}\sqrt{\frac{4\Upsilon_3^3+125\Upsilon_1\Upsilon_4^2}{\Upsilon_3\Upsilon_4}}(x-vt)+\xi_0\right)-2\Upsilon_3^2-5A_0\Upsilon_3\Upsilon_4}\right]^{\frac{1}{3}} \times \exp\left[i\left(\left(\frac{v-\alpha_1}{2a_1\lambda_6}\right)(x-vt)-\omega t\right)\right], \tag{213}$$

$$v(x,t) = \lambda_6 u(x,t), \tag{214}$$

**Table 1.** Parameter values picked for Figs. 1–3

| Parameter | Figure 1 | Figure 2 | Figure 3 |
|---|---|---|---|
| $a_1$ | −0.67 | −1.83 | −2.47 |
| $a_2$ | −0.58 | −1.11 | −1.95 |
| $e_1$ | 0.86 | NA | NA |
| $g_1$ | 0.34 | NA | 0.74 |
| $h_1$ | 0.21 | NA | NA |
| $k$ | 0.98 | 0.65 | 0.46 |
| $v$ | 1.6 | 1.79 | 2.2 |
| $\alpha_1$ | 0.46 | 0.34 | 0.41 |
| $\alpha_2$ | 0.62 | 0.93 | 0.69 |
| $\beta_1$ | 0.17 | 0.4 | NA |
| $\lambda_1$ | 0.87 | NA | NA |
| $\xi_0$ | 0.13 | 0.37 | 0.52 |
| $\omega$ | 0.52 | 0.78 | NA |
| $q_1$ | NA | 0.58 | 0.87 |
| $r_1$ | NA | 0.82 | 0.15 |
| $\lambda_3$ | NA | 0.61 | NA |
| $\xi_1$ | NA | 0.81 | 0.63 |
| $\zeta_1$ | NA | 0.56 | 0.29 |
| $\eta_1$ | NA | 0.69 | NA |
| $l_1$ | NA | NA | 0.35 |
| $m_1$ | NA | NA | 0.98 |
| $\delta_1$ | NA | NA | 0.26 |
| $\lambda_5$ | NA | NA | 0.79 |
| $\theta_1$ | NA | NA | 0.14 |
| $A_0$ | NA | NA | 0.59 |

provided

$$\Upsilon_3\left(4\Upsilon_3^3+125\Upsilon_1\Upsilon_4^2\right)<0 \text{ and } \Upsilon_4<0. \tag{215}$$

**(III)** Eqs. (6) and (7) have the rational solutions as:

$$u(x,t)=\left[A_0-\frac{2\Upsilon_3+5A_0\Upsilon_4}{5\Upsilon_4}\left\{1+\frac{10\Upsilon_4}{(2\Upsilon_3+5A_0\Upsilon_4)\sqrt{-\Upsilon_4}\,(x-vt)-10\Upsilon_4\,(5\Upsilon_4B_{0,1}\xi_0-1)}\right\}\right]^{\frac{1}{3}} \times \exp\left[i\left(\left(\frac{v-\alpha_1}{2a_1\lambda_6}\right)(x-vt)-\omega t\right)\right], \tag{216}$$

$$v(x,t)=\lambda_6 u(x,t), \tag{217}$$

provided

$$\Upsilon_0=-\frac{8\Upsilon_3^4}{625\Upsilon_4^3},\quad \Upsilon_1=-\frac{4\Upsilon_3^3}{125\Upsilon_4^2},\quad \Upsilon_2=\frac{6\Upsilon_3^2}{25\Upsilon_4} \text{ and } \Upsilon_4<0. \tag{218}$$

The chosen parameter values for the numerical simulations of the solutions (34)–(35), (103)–(104) and (170)–(171) are indicated respectively in the following Table 1.

## 5. CONCLUSIONS

This work is about retrieving soliton solutions to fiber BGs that is modeled with KE. The three cases of the power law nonlinearity factor are studied. The integration algorithm is the extended Kudryashov's scheme. This gave way to dark and singular optical solitons to the model. Thus one inherent drawback is definitely visible with this algorithm. It fails to retrieve bright optical solitons to the model. This serves as an encouragement to handle the model from additional perspectives. The abundant worksof Kudryashov serves as a strong encouragement to this end [34–40]. Lie symmetry analysis and other integration schemes will be applied to hunt down bright optical soliton solutions. Additionally, conservation laws and other features are yet to be studied. Thus, this new model on KE has given us ample issues to address in future. Our hands are therefore full.

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The authors also declare that there is no conflict of interest.

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