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Modulation instability of solutions to the complex Ginzburg–Landau equation

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Abstract
The modulation instability of continuous-wave (CW) solutions of the complex Ginzburg–Landau equation (CGLE), with arbitrary intensity-dependent nonlinearity, is studied. The variational approach and standard linear stability analysis are used to investigate the stability of CW and to obtain the criteria for modulation stability in the general form. Analytical stability criteria are established, enabling the construction of charts of stable fixed points of the cubic–quintic CGLE. We show that the evolution of modulationally stable and unstable CWs depends on the CGLE parameters. The analytical predictions for plane wave stability are confirmed by exhaustive numerical simulations.

Keywords: complex Ginzburg–Landau equation, continuous wave, stability analysis, modulation instability

(Some figures may appear in colour only in the online journal)

1. Introduction

The complex Ginzburg–Landau equation (CGLE) describes the propagation of the envelope of the electric field E in a large variety of nonlinear media [1, 2]. The stability of solutions of CGLE is one of the most important issues for possible applications. The modulation instability, which is an inherent feature of nonlinear equations, may be in some cases prevented using perfect balance of antagonistic effects, when the gain compensates losses, and the nonlinearity-induced self-contraction inhibits linear diffraction and/or dispersion. Usually, a continuous wave is unstable because of the self-phase modulation due to the cubic nonlinearity, so that the quintic nonlinearity is utilized for its stabilization [3, 4]. A catastrophic collapse is prevented by the quintic nonlinearity of the opposite sign, while the nonlinear gain compensates losses [5, 6]. Recently, higher-order conservative nonlinearities with alternative signs have been studied as stabilizing factors in the context of Schrödinger equation [7–9].

Here, we establish the stability criterion for plane waves described by the CGLE with arbitrary-order conservative and nonconservative (dissipative) nonlinearities:

$$i\frac{\partial E}{\partial z} + \sigma\frac{\partial^2 E}{\partial x^2} + F\left(\left|E\right|^2\right)E = iG\left(\left|E\right|^2\right)E + i\beta\frac{\partial^2 E}{\partial x^2}, \quad (1)$$

where the functions F and G stand for the nonlinearity and gain (or loss) in the system, while β characterizes the electric field diffusion. Such a (1 + 1)-dimensional CGLE has transverse coordinate x and propagates along the z axis. When x stands for time, the second term in equation (1) corresponds to wave dispersion, that can be either normal ($\sigma = -1$) or anomalous ($\sigma = 1$). An exact plane-wave (PW) solution with a constant amplitude A reads as follows:

$$E_0 = A\exp\left(-iqx + i\Omega z\right). \quad (2)$$

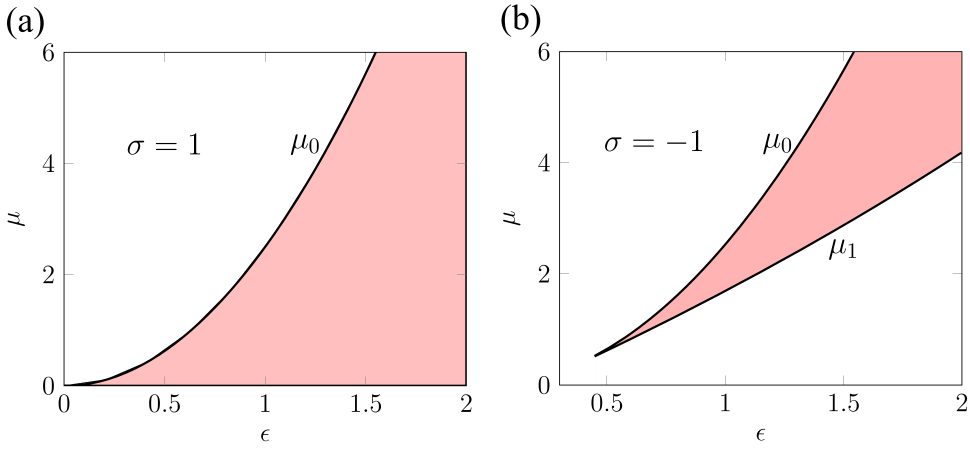

Figure 1. Regions of stability (shaded areas) of the solution A_+ for (a) anomalous and (b) normal dispersion. Fixed parameters are $\beta = 0.1$ and $\delta = 0.1$.

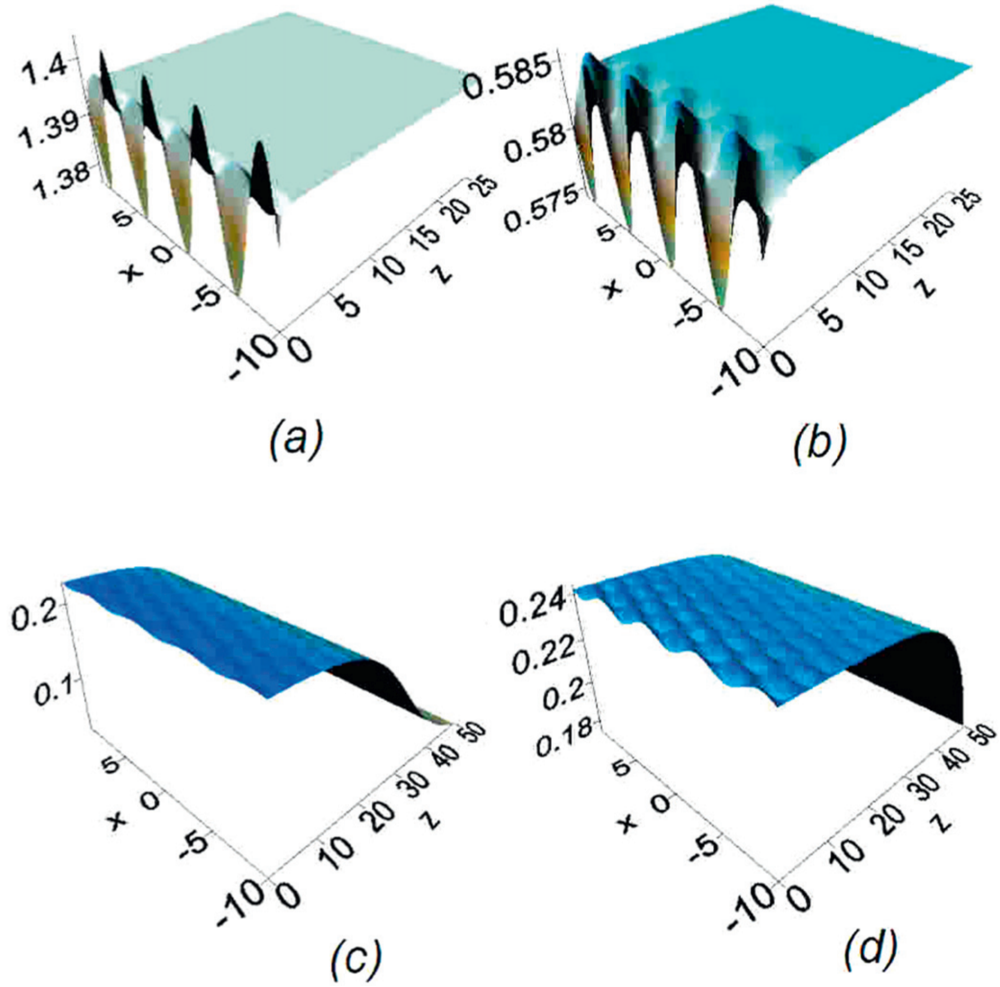

Figure 2. Typical evolution of the CW solution A_+ (upper row) and A_- (lower row) in the case of anomalous dispersion ($\sigma = 1$), for parameters $\epsilon = 2$, $\mu = 2$ (a), (c); and $\mu = 4$ (b), (d). Other parameters are $\beta = 0.1$ and $\delta = 0.1$.

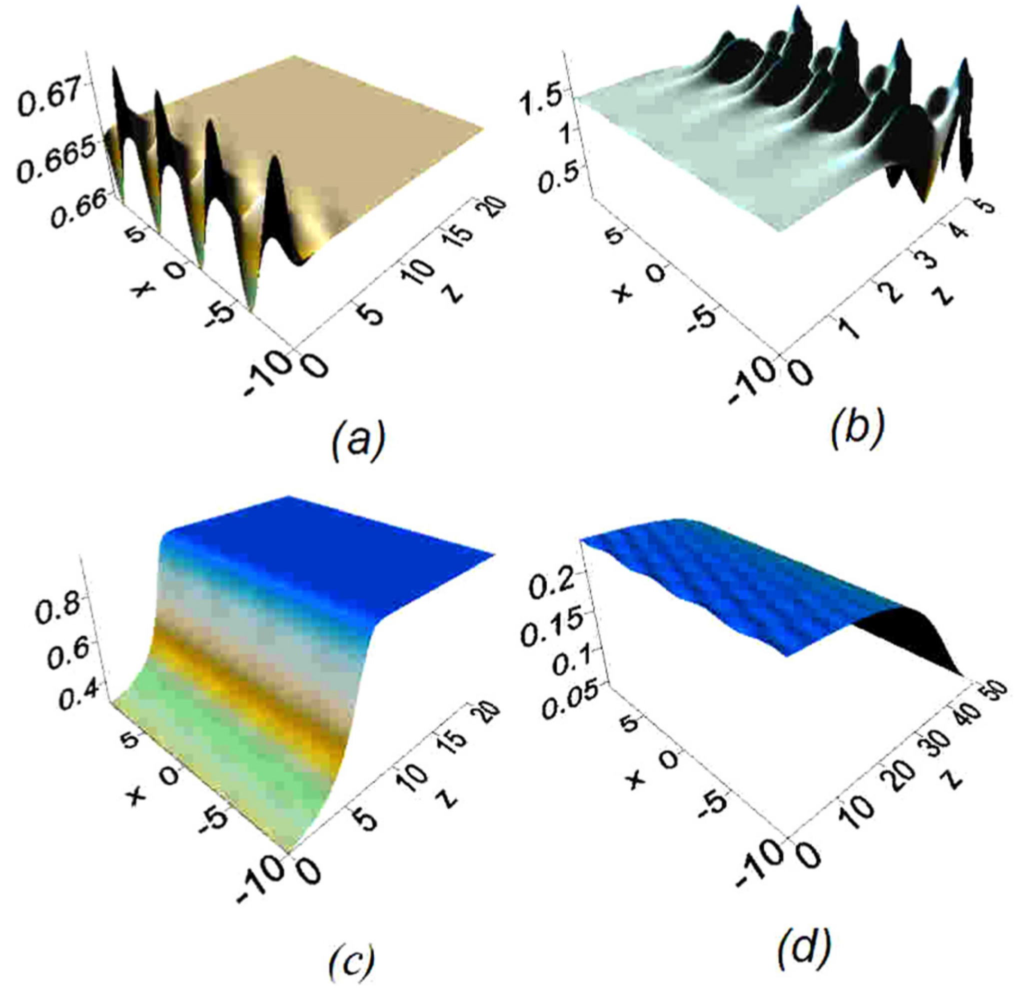

Figure 3. Evolution of the CW solution A_+ (upper row) and A_- (lower row) in the case of normal dispersion ($\sigma = -1$), for parameters $\epsilon = 2$, $\mu = 5$ (a), (c); and $\mu = 1$ (b), (d). Other parameters are $\beta = 0.1$ and $\delta = 0.1$.

The transverse wave number q and the propagation constant Ω, respectively satisfy the equations:

$$G\left(A^2\right) = \beta q^2, \tag{3}$$

and

$$\Omega = -\sigma q^2 + F\left(A^2\right). \tag{4}$$

In principle, such a solution could be stable or unstable. To study the conditions for the stabilization of the PW solution of equation (1), a small complex perturbation $P(z, x)$ is added to the amplitude, so that the perturbed electric field becomes $E = (A + P)\exp(-iqx + i\Omega z)$. After performing linearization of CGLE with the perturbed electric field, one obtains:

$$i\frac{\partial P}{\partial z} + \sigma\frac{\partial^2 P}{\partial x^2} + 2\beta q\frac{\partial P}{\partial x} + A^2 F'\left(A^2\right)\left(P + P^*\right) = iA^2 G'\left(A^2\right)\left(P + P^*\right) + i\beta\frac{\partial^2 P}{\partial x^2} - 2iq\frac{\partial P}{\partial x}, \tag{5}$$

where asterisk denotes the complex conjugation and the prime denotes the derivative,

$$F'(A) = \partial F(A)/\partial A$$

and

$$G'(A) = \partial G(A)/\partial A.$$

2. Linear stability analysis

In this paper we restrict stability analysis to a PW with $q = 0$, which is also called the continuous wave (CW). We take the perturbation P in the form $P = U(z)\sin(kx)$, with the complex amplitude $U(z) = U_R(z) + iU_I(z)$ that depends on z. To find an evolution equation for $U(z)$ we use the variational approach (VA), extended to include dissipation [5, 10]. Applying VA, the following system of linear Euler–Lagrange equations is obtained:

$$\frac{\partial U_R}{\partial z} = \left[-\beta k^2 + 2A^2 G'\left(A^2\right) \right] U_R + \sigma k^2 U_I, \tag{6}$$

and

$$\frac{\partial U_I}{\partial z} = \left[-\sigma k^2 + 2A^2 F'\left(A^2\right) \right] U_R - \beta k^2 U_I. \tag{7}$$

The eigenvalues of the Jacobian corresponding to the right-hand sides of equations (6), (7) are given by the following quadratic equation:

$$\Gamma^2 + \alpha_1 \Gamma + \alpha_2 = 0, \tag{8}$$

where the coefficients are $\alpha_1 = 2\left[\beta k^2 - A^2 G'\left(A^2\right)\right]$ and $\alpha_2 = k^4\left(1 + \beta^2\right) - 2k^2 A^2\left[\sigma F'\left(A^2\right) + \beta G'\left(A^2\right)\right]$. Note that the same result is obtained if the standard linear stability analysis is used with the perturbation in the form $P = V(z)\, exp\left[ikx\right] + W(z)\, exp\left[-ikx\right]$. Following Lyapunov, the steady-state solutions are stable if the real part of Γ is negative [5]. Therefore, the Hurwitz's conditions are:

$$G'\left(A^2\right) < 0, \tag{9}$$

and

$$\sigma F'\left(A^2\right) + \beta G'\left(A^2\right) < 0. \tag{10}$$

These inequalities, together with equation (3), represent the general form of the modulational stability conditions for equation (1). As a concrete example of these stability criteria, we consider the cubic–quintic CGLE with $F\left(A^2\right) = A^2 - A^4$ and $G\left(A^2\right) = -\delta + \epsilon A^2 - \mu A^4$, where δ, ϵ and μ are suitably introduced real parameters. The solution of equation (3),

$$A_\pm^2 = \frac{\epsilon \pm \sqrt{\epsilon^2 - 4\mu\left(\delta + \beta q^2\right)}}{2\mu}, \tag{11}$$

has two branches and it exists only if $\mu < \mu_0 = \epsilon^2/4\delta$. If $\mu > \mu_0$ then equation (1) has only the trivial solution $A = 0$.

The first Hurwitz condition, equation (9), is fulfilled ($\alpha_1 > 0$) only for the solution A_+^2. The second Hurwitz condition, equation (10), is satisfied everywhere for the solution A_+ only when $\sigma = +1$. For the continuous perturbation $k = 0$, the smaller solution A_- is always unstable ($\alpha_1 < 0$), with the growth rate $\gamma = R_e(\gamma) = \sqrt{\epsilon^2 - 4\delta\mu}$.

For anomalous dispersion $\sigma = +1$ we obtain a stable CW for $\mu_0 > \mu > 0$, together with $\epsilon > 0$. In the case of normal dispersion, $\sigma = -1$, CW is stable only if $\epsilon > \epsilon_c = 4\delta$ and $\mu_1 < \mu < \mu_0$, where

$$\mu_1 = \frac{(\beta\epsilon + 1)\left[\beta\epsilon - 1 + \sqrt{(\beta\epsilon - 1)^2 + 16\beta\delta}\right] - 8\beta\delta}{8\beta^2\delta}. \tag{12}$$

The regions of stability of the CW solution A_+ for various dissipative coefficients ϵ and μ, characterizing the cubic gain and quintic loss, are shown in figures 1(a) and (b).

3. Numerical simulation

To complete the study of stability of the CW evolution, it is necessary to make comparison between the analytically obtained stability domains and the ones resulting from the systematic numerical simulation. To this end, we implement the finite difference time domain method, extended and adapted for CGLE [11]. Exhaustive numerical simulations confirm analytically predicted stability regions from figure 1. Indeed, the stability regions obtained by analytical and numerical computations coincide.

Typical evolutions of the CW solution are demonstrated in figure 2 for the anomalous dispersion ($\sigma = 1$) and in figure 3 for the normal dispersion ($\sigma = -1$). For the nonlinear loss parameter $\mu = 2$ and $\mu = 4$, the A_+ solutions belong to the stable domain in figure 1(a). Therefore, whenever such solutions are perturbed, the perturbations quickly disappear during evolution, as can be seen in figures 2(a) and (b). In both cases the A_- solution is unstable, evolving into the trivial solution. In figures 2(c) and (d) the nonzero amplitude is vanishing after 50 propagation steps. Also in the case of normal dispersion for parameters $\epsilon = 2$ and $\mu = 1$ (see figure 3(d)) the A_- solution goes rapidly to zero. However, the same solution for a larger nonlinear loss parameter ($\mu = 5$) evolves toward the stable A_+ solution, with the value 0.685. This is the same value as the one reached directly by the A_+ solution after some oscillations, due to the initial perturbation (see figure 3(a)). In contrast, for the small nonlinear loss parameter ($\mu = 1$), the A_+ solution is out of stability region. Consequently, the initial perturbation is amplified, leading to the exponential growth, confirming analytical predictions. Such a perturbation grows in the modulationally unstable region, leading to a pattern formation of the initial CW (figure 3(b)).

4. Conclusion

Using the linear stability analysis, general analytical stability criteria are established for the CGLE with arbitrary nonlinearities. This general result is illustrated for the CW of the cubic–quintic CGLE. Based on the obtained stability criteria, a chart of the stable fixed points for various dissipative coefficients is drawn. Such analytical results are confirmed by numerical simulations of the CQ CGLE, allowing comparison of numerically and analytically obtained domains of stability for both the normal and anomalous dispersion.

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