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Traveling wave and soliton solutions of coupled nonlinear Schrödinger equations with harmonic potential and variable coefficients

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Exact traveling wave and soliton solutions, including the bright-bright and dark-dark soliton pairs, are found for the system of two coupled nonlinear Schrödinger equations with harmonic potential and variable coefficients, by employing the homogeneous balance principle and the F-expansion technique. A kind of shape-changing soliton collision is identified in the system. The collision is essentially elastic between the two solitons with opposite velocities. Our results demonstrate that the dynamics of solitons can be controlled by selecting the diffraction, nonlinearity, and gain coefficients.

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I. INTRODUCTION

Nonlinear evolution equations with variable coefficients have become more interesting and more important for various branches of natural sciences. The nonlinear Schrödinger equation (NLSE) with variable coefficients is one of the most important generic models that have recently captured attention of many researchers in different fields of physics [1,2]. Especially, there has been much interest in the multisoliton complexes (MSCs) [3]. A MSC is a self-localized state that is a nonlinear superposition of several fundamental solitons. These objects have been studied extensively, both experimentally and theoretically [3,4]. Research on incoherent spatial solitons propagating in photorefractive materials induced new interest in MSCs [4]. In general, a MSC can be described by a set of coupled NLSEs. Various solutions to these equations, including soliton solutions and periodic solutions, have been found [5,6]. These solutions are mainly obtained using the fact that MSCs are stationary, which reduces the problem of the coupled NLSEs to a set of ordinary differential equations. Radakrishnan *et al.* have considered coupled NLSE in the form of Manakov system by the Hirota method [7].

In some special cases, like the optical wave propagation in an inhomogeneous medium having a Kerr-type nonlinear response, the corresponding coupled NLSEs in $(1+1)$ dimensions with harmonic potential and variable coefficients are of the form:

$$\begin{cases} i\dfrac{\partial u}{\partial z} + \dfrac{1}{2}\beta(z)\dfrac{\partial^2 u}{\partial x^2} + (\chi_{11}|u|^2 + \chi_{12}|v|^2)u - \mu^2(z)x^2u = i\gamma(z)u \\ i\dfrac{\partial v}{\partial z} + \dfrac{1}{2}\beta(z)\dfrac{\partial^2 v}{\partial x^2} + (\chi_{21}|u|^2 + \chi_{22}|v|^2)v - \mu^2(z)x^2v = i\gamma(z)v \end{cases}, \tag{1}$$

where u and v are the components of the complex envelope of the wave, z is the normalized distance of propagation, and x is the normalized transverse coordinate. β is the diffraction (dispersion) coefficient; χ_{11}, χ_{12}, χ_{21}, and χ_{22} represent the coefficients of the inhomogeneity of the Kerr nonlinearity; they can take positive (negative) values in nonlinear self-focusing (self-defocusing) media. Note that the case when all χ are 1 describes the focusing interaction between incoherent beams and that when $\chi_{11}=\chi_{22}=1$, $\chi_{12}=\chi_{21}=2$ the interaction is coherent. Since we consider a vectorial case with two components, it is presumed that all χ cannot be equal to each other. For convenience, we assume that the fourth term $\mu(z)$ in Eq. (1) is positive, originating from the inhomogeneity of the refractive index; $\gamma(z)$ is the linear loss (gain) coefficient. The same set of equations describes the evolution of two-component Bose-Einstein condensates. In this case the evolution variable z is interpreted as time, and the equations are known as the coupled Gross-Pitaevskii equations. Two-component, or two-species, Bose-Einstein condensates have been considered in a number of references [8–14].

The simplest case when all coefficients in Eqs. (1) are constant is known as the Manakov system. The inverse scattering method (ISM) [15] was used by Manakov for finding soliton solutions [16]. The ISM is a powerful tool in constructing soliton solutions, but the high-level mathematical technicality of the method makes it difficult for finding more complex solutions, in particular those of the NLSE with variable coefficients. The stability of multicomponent solitary waves was studied in [17], collisions of MSCs were also investigated and illustrated by a numerical method in [18], and Hirota's bilinear method was used to find periodic solutions of coupled NLSEs in [7,19]. Collisions of two solitons in an arbitrary number of coupled NLSEs were considered in [20].

In this paper we present a technique that could serve well for finding more general solutions in the form of MSCs. The method is based on the homogeneous balance principle and the F-expansion technique [2]. Our method is applicable when the variable coefficients of the coupled NLSE satisfy an integrability condition. Although the method is applied here to a $(1+1)$-dimensional system, it can be easily extended to multidimensional systems. Preference is given to $(1+1)$ dimensions for stability of solitons and ease of considering collisions. The existence of exact soliton pair solutions helps us to understand the collisions between solitons of opposite velocities better. The general idea is valid for any particular nonlinearity.

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TABLE I. Some interesting soliton pair solutions.

Soliton pair	Existence condition	Soliton pair intensity
Bright-bright (BB)	$\lambda<0,\ \eta<0.$	$\lvert u(z,x)\rvert^2=\lvert\lambda\rvert D_3^2 e^{2\int(\gamma-a\beta)dz}\,\mathrm{sech}^2(\theta)$, $\lvert v(z,x)\rvert^2=\lvert\eta\rvert D_3^2 e^{2\int(\gamma-a\beta)dz}\,\mathrm{sech}^2(\theta)$.
Dark-dark (DD)	$\lambda>0,\ \eta>0.$	$\lvert u(z,x)\rvert^2=\lvert\lambda\rvert D_3^2 e^{2\int(\gamma-a\beta)dz}\,\tanh^2(\theta)$, $\lvert v(z,x)\rvert^2=\lvert\eta\rvert D_3^2 e^{2\int(\gamma-a\beta)dz}\,\tanh^2(\theta)$.

The paper is organized as follows. Section II describes the homogeneous balance principle and the F-expansion technique for the coupled NLSEs with variable coefficients, and introduces exact traveling wave and soliton pair solutions of the same. Section III is devoted to analysis of shape-changing collisions exhibited by these soliton pair solutions when variable coefficients are altered. The final section is reserved for conclusions.

II. SOLITON SOLUTIONS

To start with, we make the complex transformation $\binom{u}{v}=A_j e^{iB_j}$ $(j=1,2)$ in Eqs. (1). This results in the following set of equations for the real amplitudes A_j and the real phases B_j:

$$\frac{\partial A_j}{z}+\frac{1}{2}\beta\left(2\frac{\partial A_j}{\partial x}\frac{\partial B_j}{\partial x}+A_j\frac{\partial^2 B_j}{\partial x^2}\right)=\gamma A_j, \tag{2a}$$

$$-A_1\frac{\partial B_1}{\partial z}+\frac{1}{2}\beta\left[\frac{\partial^2 A_1}{\partial x^2}-A_1\left(\frac{\partial B_1}{\partial x}\right)^2\right]+(\chi_{11}A_1^2+\chi_{12}A_2^2)A_1 - \mu^2x^2A_1=0, \tag{2b}$$

$$-A_2\frac{\partial B_2}{\partial z}+\frac{1}{2}\beta\left[\frac{\partial^2 A_2}{\partial x^2}-A_2\left(\frac{\partial B_2}{\partial x}\right)^2\right]+(\chi_{21}A_1^2+\chi_{22}A_2^2)A_2 - \mu^2x^2A_2=0, \tag{2c}$$

By setting $A_j=f_j(z)F(\theta)+g_j(z)F^{-1}(\theta)$, $\theta=k(z)x+\omega(z)$, $B_j=a_j(z)x^2+b_j(z)x+e_j(z)$ and substituting into Eqs. (2), we derive a set of over-determined ordinary differential equations for the parameter functions f_j, g_j, k, ω, a_j, b_j, and e_j. $F(\theta)$ is a Jacobi elliptic function (JEF), satisfying the following nonlinear ordinary differential equation $(\frac{dF}{d\theta})^2=c_0+c_2F^2+c_4F^4$, where c_0, c_2, and c_4 are real constants related to the elliptic modulus of JEFs. Solving the equations for the parameter functions self-consistently, one obtains exact periodic traveling wave solutions of Eqs. (1),

$$u(z,x)=D_3\sqrt{|\lambda|}e^{\int(\gamma-a\beta)dz}\left[\sqrt{c_4}F(\theta)+\varepsilon\sqrt{c_0}\frac{1}{F(\theta)}\right]e^{iB_1}, \tag{3a}$$

$$v(z,x)=D_3\sqrt{|\eta|}e^{\int(\gamma-a\beta)dz}\left[\sqrt{c_4}F(\theta)+\varepsilon\sqrt{c_0}\frac{1}{F(\theta)}\right]e^{iB_2}, \tag{3b}$$

where $\lambda=\chi_{22}-\chi_{12}$, $\eta=\chi_{11}-\chi_{21}$, $\theta=D_1e^{-2\int\beta a dz}x\pm D_1D_2\times\int\beta e^{-4\int\beta a dz}dz+D_4$, and $B_j=ax^2+D_2e^{-2\int\beta a dz}x+\frac{1}{2}\int\beta(c_2D_1^2-D_2^2)e^{-4\int\beta a dz}dz+3D_3^2\sqrt{c_0c_4}\int(\chi_{11}\chi_{22}-\chi_{12}\chi_{21})e^{2\int(\gamma-a\beta)dz}dz+D_{4+j}$. Here D_1, D_2, D_3, D_4, D_5, and D_6 are arbitrary constants and the parameter ε can have the values $\varepsilon=0,\pm1$. Note the importance of the parameter $a(z)=a_1(z)=a_2(z)$, known as the chirp function, which figures explicitly in both the amplitudes and the phases of the component solutions. The chirp function satisfies a Ricatti-type equation of the form

$$\frac{da}{dz}+2a^2\beta+\mu^2=0, \tag{4a}$$

which requires separate attention [21,22]. As it is well known, Ricatti equation cannot be solved analytically for arbitrary coefficients $\beta(z)$ and $\mu(z)$, however its numerical solution entails little difficulty. At the same time, for the existence of exact soliton pair solutions the following constraint must be satisfied,

$$\beta D_1^2+(\chi_{11}\chi_{22}-\chi_{12}\chi_{21})D_3^2e^{2\int(\gamma+a\beta)dz}=0. \tag{4b}$$

Hence the diffraction coefficient $\beta(z)$, the determinant of the χ coefficients, and the gain $\gamma(z)$ cannot be arbitrary simultaneously.

It is of interest to determine the solitary wave solutions of Eqs. (1). They are found when the elliptic modulus of JEFs equals 1 (see references [2]); the periodic traveling wave solutions then become the spatial soliton solutions. In Table I we exhibit some interesting soliton pair intensities for $\varepsilon=0$.

III. COLLISION BETWEEN SOLITONS

We look in some detail into the collision between soliton pairs. Interactions between solitons are fascinating, since in many aspects solitons interact like particles: they pass through each other without change (except for a possible phase shift) and accomplish elastic collision. As shown in Table I, the nonlinearity coefficients appear through the pair intensity $I=(|u|^2+|v|^2)$. To better understand the collision between soliton pairs, we will follow the interaction behavior between solitons in transmission. In order to ensure that $u(z,x)$ and $v(z,x)$ are nonzero, we select appropriate nonlinearity coefficients χ_{12}, χ_{21}, χ_{11}, and χ_{22}.

First, we discuss the case $\mu=0$. Equation (1) simplifies to the standard NLSE with variable coefficients. Figure 1 shows the interaction between two solitons with opposite transverse velocities. As it is seen from Figs. 1(a) and 1(b), if we select the nonlinearity coefficients and the diffraction coefficient as periodic functions, the soliton components will collide periodically. It is shown, despite the existence of loss that the shape and the width of the two colliding solitons do not

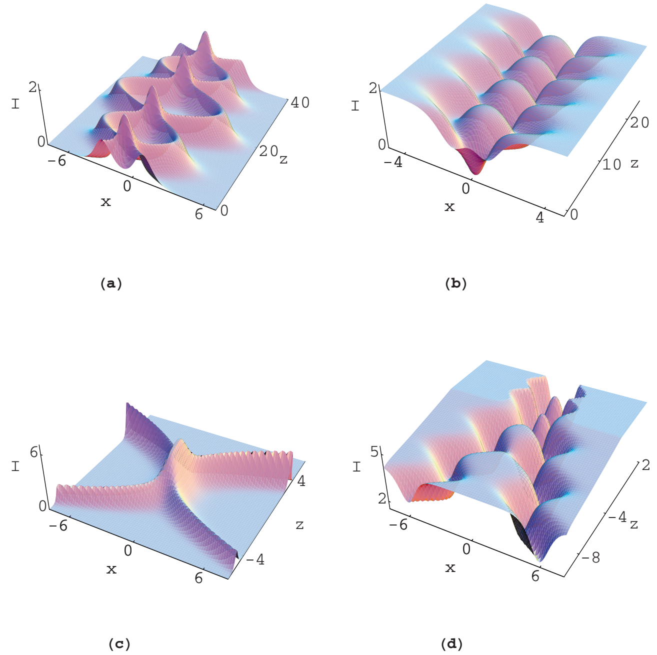

FIG. 1. (Color online) Collision between solitons of opposite velocities. $D_1=D_2=D_3=1$, $D_4=0$, the other parameters are given as follows: (a) BB soliton, $\chi_{11}=\chi_{22}=0$, $\chi_{12}=\chi_{21}=\cos(z/4)$, $\beta=\cos(z/4)$; (b) DD soliton, $\chi_{11}=\chi_{22}=0$, $\chi_{12}=\chi_{21}=\cos(z/2)$, $\beta=\cos(z/2)$; (c) BB soliton, $\chi_{11}=\chi_{22}=1$, $\chi_{12}=\chi_{21}=-2$, $\beta=1+z$; (d) DD soliton, $\chi_{11}=\chi_{22}=1$, $\chi_{12}=\chi_{21}=-2$, $\beta=-1/(1+z)^2$. In (a) and (b) $\gamma=-0.05$, in (c) and (d) $\gamma=0.05$.

change after collision, which indicates no energy exchange between them. For the constant nonlinearity coefficients and nonperiodic diffraction coefficient, the gain $(\gamma>0)$ makes the soliton compressed and the soliton amplitude increased [see Figs. 1(c) and 1(d)] along the propagation distance.

Next, we consider the case $\beta=1$ and the loss $\gamma=0$. From Eq. (4a) we can find the chirp parameter $a=\frac{\cos(\sqrt{2}\mu z)}{\sqrt{2}\mu}$. The soliton collision process corresponding to the condition (4b) is shown in Fig. 2 for the choice of parameters $\chi_{12}=\chi_{21}=1$, $\chi_{11}=\chi_{22}=2$, $D_1=D_2=D_3=1$, $D_4=0$. The figure depicts the interaction of two soliton components u and v which are well separated before and after the collision. This figure shows that it is an elastic collision. From $z=-10$, two smaller amplitude solitons approach each other, at $z=0$ they collide with each other and form a large amplitude soliton. They then separate after the collision, to restore the original shape and the width of the two solitons; there is no energy exchange during the collision.

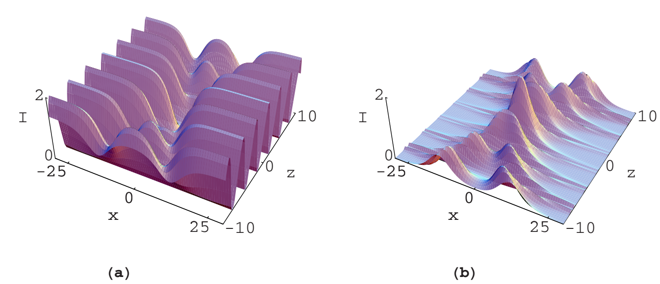

FIG. 2. (Color online) The shape-unchanging collision of two soliton components with opposite velocities. The parameters are given in the text. (a) DD soliton; (b) BB soliton.

IV. CONCLUSIONS

In this paper, we have obtained exact periodic traveling wave solutions of the coupled NLS equations with harmonic potential and variable coefficients, using the homogeneous balance principle and the F-expansion technique. Soliton pair solutions of the BB and DD type are found. We have investigated the dynamics of collisions between two soliton components with the opposite velocities. The results show that the soliton shape can be controlled by selecting the diffraction, nonlinearity, and the loss (gain) coefficients, and that the collisions are essentially elastic.

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