Original research article

Embedded solitons with $\chi^{(2)}$ and $\chi^{(3)}$ nonlinear susceptibilities by extended trial equation method

Abdullah Sonmezoglu$^a$, Mehmet Ekici$^a$, Ahmed H. Arnous$^b$, Qin Zhou$^c,*,$, Houria Triki$^d$, Seithuti P. Moshokoa$^e$, Malik Zaka Ullah$^f$, Anjan Biswas$^{e,f}$, Milivoj Belic$^g$

$^a$ Department of Mathematics, Faculty of Science and Arts, Bozok University, 66100 Yozgat, Turkey  
$^b$ Department of Engineering Mathematics and Physics, Higher Institute of Engineering, El-Shorouk, Cairo, Egypt  
$^c$ School of Electronics and Information Engineering, Wuhan Donghu University, Wuhan 430212, PR China  
$^d$ Radiation Physics Laboratory, Department of Physics, Faculty of Sciences, Badji Mokhtar University, P. O. Box 12, 23000 Annaba, Algeria  
$^e$ Department of Mathematics and Statistics, Tshwane University of Technology, Pretoria 0008, South Africa  
$^f$ Operator Theory and Applications Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, PO Box-80203, Jeddah 21589, Saudi Arabia  
$^g$ Science Program, Texas A & M University at Qatar, PO Box 23874, Doha, Qatar

ARTICLE INFO

Article history:  
Received 24 June 2017  
Accepted 3 October 2017

Keywords:  
Solitons  
$\chi^{(2)}$ and $\chi^{(3)}$ nonlinearities  
Extended trial function approach

ABSTRACT

This paper employs the extended trial equations algorithm to extract soliton and other forms of waves in quadratic-cubic nonlinear medium. These waves stem from the continuous spectrum. The solutions appear with their corresponding constraints, also known as the existence criteria for the waves.

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1. Introduction

There are several mathematical methods and tools to study various forms of nonlinear evolution equations that appear in optics and other areas of mathematical physics [1–10]. A few of these are mapping method, variational principle, Lie symmetry analysis, Kudryashov’s method and several others. This paper will employ one such integration scheme, namely the extended trial function scheme to address a fairly new problem from nonlinear optics. It is the study of embedded solitons that appear with quadratic-cubic nonlinearity. These embedded solitons appear in the continuous regime of the spectrum and is therefore less visible in this area of research. The study of such solitons with quadratic-cubic nonlinearity first appeared during 2008 [7]. Later, this problem was further studied by the aid of Lie symmetry analysis and mapping methods to extract various kinds of solitons along with the conservation law [9]. This paper serves as a sequel to previously reported results and is therefore going to focus on the application of extended trial equation method to retrieve embedded solitons. The details are discussed in the subsequent sections.

* Corresponding author.

E-mail addresses: qinzhou@whu.edu.cn, qzh@whu.edu.cn (Q. Zhou).

https://doi.org/10.1016/j.ijleo.2017.10.014  
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1. Governing equations

The governing equation for solitons in quadratic nonlinear media is given by [7,9]:

\[ \begin{align*}
    iq_t + a_1 q_{xx} + b_1 q_{xt} + c_1 q^2 r + d_1 q|q|^2 = 0, \\
    ir_t + a_2 r_{xx} + b_2 r_{xt} + c_2 r + d_2 q^2 + \delta|q|^2 r = 0.
\end{align*} \]

In Eqs. (1) and (2), the dependent variables are \( q(x, t) \) and \( r(x, t) \) which are complex valued functions representing fundamental harmonic (FH) and second harmonic (SH), respectively. The independent variables \( x \) and \( t \) are spatial and temporal variables, respectively. The coefficients \( a_1 \) and \( b_1 \) are from group velocity dispersion and spatio-temporal dispersion, respectively. Then, \( c_1 \) stands for group-velocity mismatch because of frequency difference between FH and SH fields.

2. Mathematical analysis

The extended trial function approach [2–6,10] is one of the most popular modern approaches to obtain the solutions to such NLEEs since its first appearance a few years ago. In fact, this approach has been successfully implemented to extract soliton solutions to water wave model, plasma physics model as well as in nonlinear optics and nuclear physics. In order to study the details of this model, the hypothesis is [1,7,9]:

\[ \begin{align*}
    q(x, t) &= P_1(\xi)e^{i\phi(x, t)}, \\
    r(x, t) &= P_2(\xi)e^{i\phi(x, t)},
\end{align*} \]

where the wave variable \( \xi \) is given by

\[ \xi = x - vt. \]

Here, \( P_l(\xi) \) for \( l = 1, 2 \) represents the amplitude component and \( v \) is the speed of the wave, while phase is defined as

\[ \phi(x, t) = -\kappa x + \omega t + \Theta, \]

where \( \kappa \) is the soliton frequency, \( \omega \) is the soliton wave number and \( \Theta \) the phase constant. Substituting (3) and (4) into (1) and (2) and decomposing into real and imaginary parts gives

\[ \begin{align*}
    (b_1 v - a_1) P_1' + (\omega + a_1 \kappa^2 - b_1 \omega \kappa) P_1 - c_1 P_1 P_2 - d_1 P_1^2 &= 0, \\
    v &= \frac{b_1 \omega - 2a_1 \kappa}{1 - b_1 \kappa},
\end{align*} \]

respectively, from the first component. Similarly, the second component respectively gives

\[ \begin{align*}
    (b_2 v - a_2) P_2' + (2\omega + 4a_2 \kappa^2 - 4b_2 \omega \kappa - c_2) P_2 - d_2 P_2^2 - \delta P_1^2 P_2 &= 0, \\
    v &= \frac{2b_2 \omega - 4a_2 \kappa}{1 - 2b_2 \kappa}.
\end{align*} \]

From (8) and (10) equating the two values of the speed \( v \) leads to

\[ a_1 = 2a_2, \]

and

\[ b_1 = 2b_2. \]

Thus, the governing equations modify to

\[ \begin{align*}
    iq_t + 2a_1 q_{xx} + 2b_1 q_{xt} + c_1 q^2 r + d_1 q|q|^2 = 0, \\
    ir_t + a_2 r_{xx} + b_2 r_{xt} + c_2 r + d_2 q^2 + \delta|q|^2 r = 0.
\end{align*} \]

Hence, the real part equations from the two components can be written as

\[ \begin{align*}
    2(bv - a) P_1' + (\omega + 2a \kappa^2 - 2b \omega \kappa) P_1 - c_1 P_1 P_2 - d_1 P_1^2 &= 0, \\
    (bv - a) P_2' + (2\omega + 4a \kappa^2 - 4b \omega \kappa - c_2) P_2 - d_2 P_2^2 - \delta P_1^2 P_2 &= 0.
\end{align*} \]

These two equations will be further studied in the next subsection to retrieve solitons and other forms of wave solutions.
2.1. Extended trial equation scheme

To start with the extraction of solutions to (15) and (16), the following assumption for the solution structure is hypothesized:

\[
P_1 = \sum_{i=0}^{\zeta} \tau_i \Psi^i,
\]

\[
P_2 = \sum_{i=0}^{\zeta} \bar{\tau}_i \Psi^i,
\]

where

\[
(\Psi')^2 = \Lambda(\Psi) = \frac{\Phi(\Psi)}{Y(\Psi)} = \frac{\mu_\sigma \Psi^\sigma + \cdots + \mu_1 \Psi + \mu_0}{\chi_\rho \Psi^\rho + \cdots + \chi_1 \Psi + \chi_0}.
\]

Here \(\tau_0, \ldots, \tau_\zeta; \bar{\tau}_0, \ldots, \bar{\tau}_\zeta; \mu_0, \ldots, \mu_\sigma\) and \(\chi_0, \ldots, \chi_\rho\) are constants to be determined later. Eq. (19) can be reduced to the integral form given by

\[
\pm(\zeta - \zeta_0) = \int \frac{d\Psi}{\sqrt{\Lambda(\Psi)}} = \int \sqrt{\frac{\Upsilon(\Psi)}{\Phi(\Psi)}} d\Psi.
\]

According to the balance principle, we procure a relation of \(\sigma, \rho, \zeta\) and \(\bar{\zeta}\) as

\[
\bar{\zeta} = 2\zeta = \sigma - \rho - 2.
\]

Let us choose \(\sigma=4\) and \(\rho=0\) in (21), then \(\zeta=1\) and \(\bar{\zeta}=2\). Therefore, Eqs. (15) and (16) have formal solutions in the forms:

\[
P_1 = \tau_0 + \tau_1 \Psi,
\]

\[
P_2 = \bar{\tau}_0 + \bar{\tau}_1 \Psi + \bar{\tau}_2 \Psi^2,
\]

where \(\tau_i, \bar{\tau}_j\) for \(i=0, 1\) and \(j=0, 1, 2\) are constants to be determined later, and \(\Psi\) satisfies Eq. (19). Substituting these formal solutions into (15) and (16), and solving the resulting system of algebraic equations we achieve

\[
\mu_0 = -27c_0 \delta \mu_4 \hat{\eta}_1 - 54c_1 d_2 \mu_4 \hat{\eta}_1 + \delta (\delta - 6d_1) \left[ 3\mu_4 \tau_0^2 \left( -36 \hat{\eta}_1 - 5 \tau_0^2 (\delta - 6d_1) (2\delta - 3d_1) \right) + \mu_2 \tau_1^2 \hat{\eta}_2 \right],
\]

\[
\mu_1 = 2 \left( \mu_2 \tau_0 \tau_1 - 4 \mu_4 \tau_0^3 \right) \frac{\delta \tau_1^2 (\delta - 6d_1)^2 (2\delta - 3d_1)}{\tau_1^4},
\]

\[
\mu_3 = 4 \mu_4 \tau_0, \quad \chi_0 = -6 \mu_4 (a - bv),
\]

\[
\tau_0 = \frac{9c_1 d_2 + (2\delta - 3d_1)}{3c_1 \left( \delta - 6d_1 \right)}, \quad \tau_1 = \frac{2\tau_0 \tau_1 (2\delta - 3d_1)}{3c_1}, \quad \tau_2 = \frac{\tau_1^2 (2\delta - 3d_1)}{3c_1}.
\]

\[
\omega = \frac{3\mu_4 \left[ 2ak^2 (\delta - 6d_1) - \hat{\eta}_1 - 2\delta \tau_0^2 (\delta - 6d_1) \right] + \delta \mu_2 \tau_1^2 (\delta - 6d_1)}{3\mu_4 (2bk - 1) (\delta - 6d_1)},
\]

\[
\mu_2 = \mu_2, \quad \mu_4 = \mu_4, \quad \tau_0 = 0, \quad \tau_1 = \tau_1.
\]

where

\[
\hat{\eta}_1 = 3c_1 d_2 + c_2 (2\delta - 3d_1),
\]

\[
\hat{\eta}_2 = 18c_1 d_2 + (2\delta - 3d_1) \left[ 6c_2 + \tau_0^2 (\delta - 6d_1) \right].
\]

Substituting these results into (19) and (20) leads to

\[
\pm(\zeta - \zeta_0) = \Omega \int \frac{d\Psi}{\sqrt{\Lambda(\Psi)}},
\]

where

\[
\Lambda(\Psi) = \Psi^4 + \frac{\mu_3}{\mu_4} \Psi^3 + \frac{\mu_2}{\mu_4} \Psi^2 + \frac{\mu_1}{\mu_4} \Psi + \frac{\mu_0}{\mu_4}, \quad \Omega = \sqrt{\frac{\chi_0}{\chi_1}}.
\]

As a result, we extract traveling wave solutions to the governing model as below:
For $\Delta(\Psi) = (\Psi - \lambda_1)^4$, 

$$q(x, t) = \begin{cases} \tau_0 + \tau_1 \lambda_1 + \frac{\tau_1 \Omega}{x - \frac{2b\omega - 4ak}{1 - 2bk}} t - \zeta_0 \\ \times \exp \left[ i \left\{ -kx + \frac{3\mu_4 \left[ 2ak^2 \left( \delta - 6d_1 \right) - \tau_1 - 2\delta t_0^2 \left( \delta - 6d_1 \right) \right] + \delta \mu_2 \tau_1^2 \left( \delta - 6d_1 \right) }{3\mu_4(2bk - 1) \left( \delta - 6d_1 \right)} \right] t + \theta \right] \end{cases} \right) ,$$

(28)

$$r(x, t) = \sum_{j=0}^{2} \tilde{r}_j \left( \lambda_1 + \frac{\Omega}{x - \frac{2b\omega - 4ak}{1 - 2bk}} t - \zeta_0 \right)^j \times \exp \left[ 2i \left\{ -kx + \frac{3\mu_4 \left[ 2ak^2 \left( \delta - 6d_1 \right) - \tau_1 - 2\delta t_0^2 \left( \delta - 6d_1 \right) \right] + \delta \mu_2 \tau_1^2 \left( \delta - 6d_1 \right) }{3\mu_4(2bk - 1) \left( \delta - 6d_1 \right)} \right] t + \theta \right] \right) \right) ,$$

(29)

If $\Delta(\Psi) = (\Psi - \lambda_1)^3(\Psi - \lambda_2)$ and $\lambda_2 > \lambda_1$, 

$$q(x, t) = \begin{cases} \tau_0 + \tau_1 \lambda_1 + \frac{4\tau_1 \Omega^2(\lambda_2 - \lambda_1)}{4\Omega^2 - \left[ (\lambda_1 - \lambda_2) \left( x - \frac{2b\omega - 4ak}{1 - 2bk} \right) t - \zeta_0 \right]^2} \\ \times \exp \left[ i \left\{ -kx + \frac{3\mu_4 \left[ 2ak^2 \left( \delta - 6d_1 \right) - \tau_1 - 2\delta t_0^2 \left( \delta - 6d_1 \right) \right] + \delta \mu_2 \tau_1^2 \left( \delta - 6d_1 \right) }{3\mu_4(2bk - 1) \left( \delta - 6d_1 \right)} \right] t + \theta \right] \end{cases} \right) ,$$

(30)

$$r(x, t) = \sum_{j=0}^{2} \tilde{r}_j \left( \lambda_1 + \frac{4\Omega^2(\lambda_2 - \lambda_1)}{4\Omega^2 - \left[ (\lambda_1 - \lambda_2) \left( x - \frac{2b\omega - 4ak}{1 - 2bk} \right) t - \zeta_0 \right]^2} \right)^j \times \exp \left[ 2i \left\{ -kx + \frac{3\mu_4 \left[ 2ak^2 \left( \delta - 6d_1 \right) - \tau_1 - 2\delta t_0^2 \left( \delta - 6d_1 \right) \right] + \delta \mu_2 \tau_1^2 \left( \delta - 6d_1 \right) }{3\mu_4(2bk - 1) \left( \delta - 6d_1 \right)} \right] t + \theta \right] \right) \right) ,$$

(31)

The solutions (28)–(31) represent plane waves.

Next, when $\Delta(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)^2$, 

$$q(x, t) = \begin{cases} \tau_0 + \tau_1 \lambda_2 + \frac{\tau_1 (\lambda_2 - \lambda_1)}{\exp \left[ \frac{\lambda_1 - \lambda_2}{\Omega} \left( x - \frac{2b\omega - 4ak}{1 - 2bk} \right) t - \zeta_0 \right] - 1} \\ \times \exp \left[ i \left\{ -kx + \frac{3\mu_4 \left[ 2ak^2 \left( \delta - 6d_1 \right) - \tau_1 - 2\delta t_0^2 \left( \delta - 6d_1 \right) \right] + \delta \mu_2 \tau_1^2 \left( \delta - 6d_1 \right) }{3\mu_4(2bk - 1) \left( \delta - 6d_1 \right)} \right] t + \theta \right] \end{cases} \right) ,$$

(32)

$$r(x, t) = \sum_{j=0}^{2} \tilde{r}_j \left( \lambda_2 + \frac{\lambda_2 - \lambda_1}{\exp \left[ \frac{\lambda_1 - \lambda_2}{\Omega} \left( x - \frac{2b\omega - 4ak}{1 - 2bk} \right) t - \zeta_0 \right] - 1} \right)^j \times \exp \left[ 2i \left\{ -kx + \frac{3\mu_4 \left[ 2ak^2 \left( \delta - 6d_1 \right) - \tau_1 - 2\delta t_0^2 \left( \delta - 6d_1 \right) \right] + \delta \mu_2 \tau_1^2 \left( \delta - 6d_1 \right) }{3\mu_4(2bk - 1) \left( \delta - 6d_1 \right)} \right] t + \theta \right] \right) \right) ,$$

(33)

and

$$q(x, t) = \begin{cases} \tau_0 + \tau_1 \lambda_1 + \frac{\tau_1 (\lambda_1 - \lambda_2)}{\exp \left[ \frac{\lambda_1 - \lambda_2}{\Omega} \left( x - \frac{2b\omega - 4ak}{1 - 2bk} \right) t - \zeta_0 \right] - 1} \\ \times \exp \left[ i \left\{ -kx + \frac{3\mu_4 \left[ 2ak^2 \left( \delta - 6d_1 \right) - \tau_1 - 2\delta t_0^2 \left( \delta - 6d_1 \right) \right] + \delta \mu_2 \tau_1^2 \left( \delta - 6d_1 \right) }{3\mu_4(2bk - 1) \left( \delta - 6d_1 \right)} \right] t + \theta \right] \end{cases} \right) ,$$

(34)
\[ r(x, t) = \sum_{j=0}^{2} \left\{ \lambda_1 + \frac{\lambda_1 - \lambda_2}{\exp \left[ \frac{\lambda_1 - \lambda_2}{\Omega} \left( x - \left\{ \frac{2\omega - 4ak}{1 - 2bk} \right\} t - \zeta_0 \right) - 1 \right] } \right\}^j \times \exp \left[ 2i \left\{ -kx + \left( \frac{3\mu_4 [2ak^2 (\delta - 6d_1) - \tau_1 - 2\delta \tau_0^2 (\delta - 6d_1)] + \delta \mu_2 \tau_t^2 (\delta - 6d_1)}{3\mu_4(2\beta - 1)(\delta - 6d_1)} \right) t + \theta \right\} \right] . \]  

Whenever \( \Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)(\Psi - \lambda_3) \) and \( \lambda_1 > \lambda_2 > \lambda_3, \)

\[ q(x, t) = \left\{ \tau_0 + \tau_1 \lambda_1 - \frac{2\tau_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}{2\lambda_1 - \lambda_2 - \lambda_3 + (\lambda_3 - \lambda_2) \cosh \left[ \frac{\sqrt{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}}{\Omega} \left( x - \left\{ \frac{2\omega - 4ak}{1 - 2bk} \right\} t \right) \right] } \times \exp \left[ i \left\{ -kx + \left( \frac{3\mu_4 [2ak^2 (\delta - 6d_1) - \tau_1 - 2\delta \tau_0^2 (\delta - 6d_1)] + \delta \mu_2 \tau_t^2 (\delta - 6d_1)}{3\mu_4(2\beta - 1)(\delta - 6d_1)} \right) t + \theta \right\} \right] \right\} . \]  

On the other hand, if \( \Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)(\Psi - \lambda_3)(\Psi - \lambda_4) \) and \( \lambda_1 > \lambda_2 > \lambda_3 > \lambda_4, \)

\[ q(x, t) = \left\{ \tau_0 + \tau_1 \lambda_2 + \frac{\tau_1(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_4 - \lambda_2 + (\lambda_1 - \lambda_4) \sin^2 \left[ \pm \sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)} \left( x - \left\{ \frac{2\omega - 4ak}{1 - 2bk} \right\} t - \zeta_0 \right) \right], m \times \exp \left[ i \left\{ -kx + \left( \frac{3\mu_4 [2ak^2 (\delta - 6d_1) - \tau_1 - 2\delta \tau_0^2 (\delta - 6d_1)] + \delta \mu_2 \tau_t^2 (\delta - 6d_1)}{3\mu_4(2\beta - 1)(\delta - 6d_1)} \right) t + \theta \right\} \right] \right\} . \]  

\[ r(x, t) = \sum_{j=0}^{2} \left\{ \lambda_2 + \frac{(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_4 - \lambda_2 + (\lambda_1 - \lambda_4) \sin^2 \left[ \pm \sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)} \left( x - \left\{ \frac{2\omega - 4ak}{1 - 2bk} \right\} t - \zeta_0 \right) \right], m \times \exp \left[ 2i \left\{ -kx + \left( \frac{3\mu_4 [2ak^2 (\delta - 6d_1) - \tau_1 - 2\delta \tau_0^2 (\delta - 6d_1)] + \delta \mu_2 \tau_t^2 (\delta - 6d_1)}{3\mu_4(2\beta - 1)(\delta - 6d_1)} \right) t + \theta \right\} \right] \right\} . \]  

where modulus \( m \) is given by

\[ m^2 = \frac{(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_4)}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}. \]  

It should be noted that \( \lambda_j \) for \( j = 1, \ldots, 4 \) are the roots of equation \( \Lambda(\Psi) = 0 \).

Under the conditions \( \tau_0 = -\tau_1 \lambda_1 \) and \( \zeta_0 = 0 \), the solutions (28)-(37) can be reduced exact solutions in the following forms:
Plane wave solutions are

\[
q(x, t) = \begin{cases} 
\frac{\tau_1 \Omega}{x - \left(2 \frac{b \omega - 4a \kappa}{1 - 2b \kappa} \right) t} \\
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\lambda_1 + \frac{\Omega}{x - \left(2 \frac{b \omega - 4a \kappa}{1 - 2b \kappa} \right) t} \\
\sum_{j=0}^{2} \left( \lambda_1 + \frac{\Omega}{x - \left(2 \frac{b \omega - 4a \kappa}{1 - 2b \kappa} \right) t} \right)^j \\
\exp \left[ i \left( -kx + \frac{3 \mu_4 \left[ 2a^2 (\delta - 6d_1) - \tau_1 - 2 \delta \tau_0^2 (\delta - 6d_1) \right]}{3 \mu_4 (2b \kappa - 1) (\delta - 6d_1)} t + \theta \right) \right]
\end{cases}
\]

\[
r(x, t) = \begin{cases} 
\frac{\tau_1 \Omega^2 (\lambda_2 - \lambda_1)}{4 \Omega^2 - \left( \lambda_1 - \lambda_2 \right) \left( x - \left(2 \frac{b \omega - 4a \kappa}{1 - 2b \kappa} \right) t \right)^2} \\
\sum_{j=0}^{2} \left( \lambda_1 + \frac{4 \Omega^2 (\lambda_2 - \lambda_1)}{4 \Omega^2 - \left( \lambda_1 - \lambda_2 \right) \left( x - \left(2 \frac{b \omega - 4a \kappa}{1 - 2b \kappa} \right) t \right)^2} \right)^j \\
\exp \left[ i \left( -kx + \frac{3 \mu_4 \left[ 2a^2 (\delta - 6d_1) - \tau_1 - 2 \delta \tau_0^2 (\delta - 6d_1) \right]}{3 \mu_4 (2b \kappa - 1) (\delta - 6d_1)} t + \theta \right) \right]
\end{cases}
\]

Traveling wave solutions are

\[
q(x, t) = \begin{cases} 
\frac{\tau_1 (\lambda_2 - \lambda_1)}{2} \left( 1 + \cosh \left[ \frac{\lambda_1 - \lambda_2}{2 \Omega} \left( x - \left(2 \frac{b \omega - 4a \kappa}{1 - 2b \kappa} \right) t \right) \right] \right) \\
\sum_{j=0}^{2} \left( \lambda_1 + \frac{\lambda_2 - \lambda_1}{\Omega} \left( x - \left(2 \frac{b \omega - 4a \kappa}{1 - 2b \kappa} \right) t \right) - 1 \right)^j \\
\exp \left[ i \left( -kx + \frac{3 \mu_4 \left[ 2a^2 (\delta - 6d_1) - \tau_1 - 2 \delta \tau_0^2 (\delta - 6d_1) \right]}{3 \mu_4 (2b \kappa - 1) (\delta - 6d_1)} t + \theta \right) \right]
\end{cases}
\]

\[
r(x, t) = \begin{cases} 
\frac{\lambda_1 + \frac{\lambda_1 - \lambda_2}{\Omega} \left( x - \left(2 \frac{b \omega - 4a \kappa}{1 - 2b \kappa} \right) t \right) - 1}{\Omega} \\
\sum_{j=0}^{2} \left( \lambda_1 + \frac{\lambda_1 - \lambda_2}{\Omega} \left( x - \left(2 \frac{b \omega - 4a \kappa}{1 - 2b \kappa} \right) t \right) - 1 \right)^j \\
\exp \left[ i \left( -kx + \frac{3 \mu_4 \left[ 2a^2 (\delta - 6d_1) - \tau_1 - 2 \delta \tau_0^2 (\delta - 6d_1) \right]}{3 \mu_4 (2b \kappa - 1) (\delta - 6d_1)} t + \theta \right) \right]
\end{cases}
\]
Bright soliton solutions are

\[
q(x, t) = \left\{ \frac{M}{R + \cosh \left[ Q \left( x - \left\{ \frac{2b\omega - 4aK}{1 - 2bK} \right\} \right) \right]} \times \exp \left[ i \left\{ -kx + \left( \frac{3\mu_4 \left[ 2ak^2 \left( \delta - 6d_1 \right) - \gamma_1 - 2\delta \tau_0^2 \left( \delta - 6d_1 \right) \right] + \delta \mu_2 \tau_1^2 \left( \delta - 6d_1 \right) \right)}{3\mu_4(2bk - 1) \left( \delta - 6d_1 \right)} \right\} t + \theta \right]\}.
\]

\[ (49) \]

\[
r(x, t) = \sum_{j=0}^{2} \tilde{r}_j \left( \frac{\lambda_1 - \frac{2(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}{2\lambda_1 - \lambda_2 - \lambda_3 + (\lambda_3 - \lambda_2) \cosh \left[ Q \left( x - \left\{ \frac{2b\omega - 4aK}{1 - 2bK} \right\} \right) \right]} j}{\lambda_3 - \lambda_2} \right) \times \exp \left[ 2i \left\{ -kx + \left( \frac{3\mu_4 \left[ 2ak^2 \left( \delta - 6d_1 \right) - \gamma_1 - 2\delta \tau_0^2 \left( \delta - 6d_1 \right) \right] + \delta \mu_2 \tau_1^2 \left( \delta - 6d_1 \right) \right)}{3\mu_4(2bk - 1) \left( \delta - 6d_1 \right)} \right\} t + \theta \right]\}.
\]

\[ (50) \]

where

\[ M = \frac{2\tau_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}{\lambda_3 - \lambda_2}, \]

\[ Q = \frac{\sqrt{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}}{\Omega}, \]

\[ R = \frac{2\lambda_1 - \lambda_2 - \lambda_3}{\lambda_3 - \lambda_2}. \]

Here the amplitude of the solitons is given by (51), while the inverse width of the solitons is given by (52). These solitons are valid for \( \tau_1 < 0 \). Moreover, under the conditions \( \tau_0 = -\tau_1 \lambda_2 \) and \( \xi_0 = 0 \), Jacobi elliptic function solutions (38) and (39) are reduced to

\[
q(x, t) = \left\{ \frac{M_1}{R_1 + \text{sn}^2 \left[ Q_1 \left( x - \left\{ \frac{2b\omega - 4aK}{1 - 2bK} \right\} \right) \right]} \times \exp \left[ i \left\{ -kx + \left( \frac{3\mu_4 \left[ 2ak^2 \left( \delta - 6d_1 \right) - \gamma_1 - 2\delta \tau_0^2 \left( \delta - 6d_1 \right) \right] + \delta \mu_2 \tau_1^2 \left( \delta - 6d_1 \right) \right)}{3\mu_4(2bk - 1) \left( \delta - 6d_1 \right)} \right\} t + \theta \right]\}.
\]

\[ (54) \]

\[
r(x, t) = \sum_{j=0}^{2} \tilde{r}_j \left( \frac{(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_4 - \lambda_2 + (\lambda_1 - \lambda_4) \text{sn}^2 \left[ Q_1 \left( x - \left\{ \frac{2b\omega - 4aK}{1 - 2bK} \right\} \right) \right]} \right) \times \exp \left[ 2i \left\{ -kx + \left( \frac{3\mu_4 \left[ 2ak^2 \left( \delta - 6d_1 \right) - \gamma_1 - 2\delta \tau_0^2 \left( \delta - 6d_1 \right) \right] + \delta \mu_2 \tau_1^2 \left( \delta - 6d_1 \right) \right)}{3\mu_4(2bk - 1) \left( \delta - 6d_1 \right)} \right\} t + \theta \right]\}.
\]

\[ (55) \]

where

\[ M_1 = \frac{\tau_1(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_1 - \lambda_4}, \]

\[ R_1 = \frac{\lambda_4 - \lambda_2}{\lambda_1 - \lambda_4}, \]

\[ Q_1 = \frac{(-1)^j \sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}}{2\Omega} \text{ for } j = 1, 2. \]

\[ (56) \]

Remark 1. When the modulus \( m \to 1 \), singular optical soliton solutions are acquired:

\[
q(x, t) = \left\{ \frac{M_1}{R_1 + \text{th}^2 \left[ Q_1 \left( x - \left\{ \frac{2b\omega - 4aK}{1 - 2bK} \right\} \right) \right]} \times \exp \left[ i \left\{ -kx + \left( \frac{3\mu_4 \left[ 2ak^2 \left( \delta - 6d_1 \right) - \gamma_1 - 2\delta \tau_0^2 \left( \delta - 6d_1 \right) \right] + \delta \mu_2 \tau_1^2 \left( \delta - 6d_1 \right) \right)}{3\mu_4(2bk - 1) \left( \delta - 6d_1 \right)} \right\} t + \theta \right]\}.
\]

\[ (59) \]
3. Conclusions

This paper secured bright and singular embedded solitons for quadratic–cubic nonlinear medium. The powerful extended trial function scheme made it possible to obtain these solutions. In addition, as a byproduct of this integration scheme, plane waves, periodic singular waves as well as elliptic function solution were also retrieved. In the limiting case, these elliptic functions yielded singular soliton solutions. The results of this paper are thus overwhelming and are very meaningful in such a medium. The results of this paper thus lead to several avenues of further future research in this direction. Some of them are the consideration with fractional temporal evolution, perturbed versions of the model, study of the model with time-dependent coefficients or stochastic coefficients. The outcome of such research are under way and their respective results will be sequentially disseminated.

Conflict of interest

The authors also declare that there is no conflict of interest.

Acknowledgements

The fourth author (QZ) was funded by the National Natural Science Foundation of China under the grant number 11705130. The sixth author (SPM) would like to thank the research support provided by the Department of Mathematics and Statistics at Tshwane University of Technology and the support from the South African National Foundation under Grant Number 92052 IRF1202210126. The research work of ninth author (MB) was supported by Qatar National Research Fund (QNRF) under the grant number NPRP 8-028-1-001.

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