Exact solution to four-wave mixing with complex couplings: reflection geometry

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An exact solution to photorefractive four-wave mixing equations with complex couplings in reflection geometry is obtained. It is shown that the efficiency of the process of phase conjugation can be enhanced by introduction of a frequency shift between the pumps and the signal, similar to the case of transmission geometry. However, to obtain an improved agreement with experiment, the inclusion of transverse effects is found to be necessary.

In a seminal paper[1] published in 1982, Cronin-Golomb *et al.* obtained an exact solution to the problem of four-wave mixing in the transmission geometry (TG), in which the coupling constant γ is allowed to be an arbitrary complex number. Simplified procedures[2] soon followed that could treat the TG or the reflection geometry (RG) of wave mixing in photorefractive (PR) crystals for γ real. This amounts to assuming that the buildup of space-charge field in the crystal is dominated by the diffusion of photoexcited charge carriers. We[3] introduced a unified procedure that treats both the TG and the RG on an equal footing, also for γ real. A natural question is whether there exists a unified procedure for γ complex. Such a choice for γ is relevant for many physically interesting situations, for example, when one applies an external electric field to the crystal (to enhance wave coupling) or when one encounters or enforces detuning of intracavity fields in PR oscillators (to maximize reflectivity).

The short answer to the question posed is in the affirmative; however, there remains the problem of the most effective presentation of such a procedure. Owing to symmetries[3,4] that exist between TG and RG, a procedure invented to solve one case (TG) usually leads to a procedure to solve the other (RG). However, owing to symmetry-breaking mechanisms built into the physics of these two wave-mixing processes, new procedures usually are less practical or comprehensible. Thus we try to devise a simple procedure for γ complex. In our opinion, the most effective approach is to generalize directly the method of Cronin-Golomb *et al.* to include RG. Curiously enough, Cronin-Golomb *et al.* choose not to generalize their TG method to treat RG but instead developed an entirely different method,[2] which was suited only for the case of γ real. In the end, we apply the procedure to the problem of spontaneous frequency detunings. This problem, as it turned out, can be resolved only if transverse effects are taken into account.

The object is to solve four-wave mixing phase-conjugation (PC) equations[2]:

$$\frac{\mathrm{d}A_1}{\mathrm{d}z} = \frac{\gamma}{I} Q A_3, \qquad \frac{\mathrm{d}A_2^*}{\mathrm{d}z} = \frac{\gamma}{I} Q A_4^*, \tag{1a}$$

$$\frac{\mathrm{d}A_3^*}{\mathrm{d}z} = \frac{\gamma}{I} Q A_1^*, \qquad \frac{\mathrm{d}A_4}{\mathrm{d}z} = \frac{\gamma}{I} Q A_2, \tag{1b}$$

in the standard RG arrangement. Beams A_1 and A_4 illuminate the crystal from the $z = 0$ side, with the boundary conditions $A_{1,4}(0) = C_{1,4}$. Beams A_2 and A_3 illuminate the crystal from the $z = d$ side, with the boundary conditions $A_{2,3}(d) = C_{2,3}$, where d is the thickness of the crystal. The angle at the beams' intersection is assumed small, and z is the propagation direction. Here $Q = A_1 A_3^* + A_2^* A_4$ is the amplitude of the grating that is generated in the crystal and I is the total intensity. The asterisk stands for complex conjugation.

The solution procedure is facilitated by the existence of conserved quantities $c = A_1 A_2 - A_3 A_4$, $d_1 = I_1 - I_3$, and $d_2 = I_2 - I_4$. With their help, Eqs. (1) are separated into a system of two equations for the ratios of waves 1 and 2 and waves 3 and 4:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{A_1}{A_2^*}\right) = -\frac{\gamma}{I}\left[c + (d_1 - d_2)\left(\frac{A_1}{A_2^*}\right) - c^*\left(\frac{A_1}{A_2^*}\right)^2\right], \tag{2a}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{A_3^*}{A_4}\right) = \frac{\gamma}{I}\left[c^* + (d_1 - d_2)\left(\frac{A_3^*}{A_4}\right) - c\left(\frac{A_3^*}{A_4}\right)^2\right]. \tag{2b}$$

These equations can be integrated, once the following transformation of the independent variable z is performed:

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = \frac{\delta}{2I}, \tag{3}$$

where $\delta = [(d_1 - d_2)^2 + 4|c|^2]^{1/2}$. The solutions are of the form

$$\frac{A_1}{A_2^*} = -\frac{(\Delta - \delta)D\exp(-\gamma\theta) - (\Delta + \delta)D^{-1}\exp(\gamma\theta)}{2c^*[D\exp(-\gamma\theta) - D^{-1}\exp(\gamma\theta)]}, \tag{4a}$$

$$\frac{A_3^*}{A_4} = \frac{(\Delta - \delta)E^{-1}\exp(\gamma\theta) - (\Delta + \delta)E\exp(-\gamma\theta)}{2c[E\exp(-\gamma\theta) - E^{-1}\exp(\gamma\theta)]}, \tag{4b}$$

where $\Delta = d_2 - d_1$, and D and E are the constants of integration, to be found from the boundary conditions. They are the functions of Δ, $|c|^2$, and θ_d (the subscript d means that the quantity is evaluated at $z = d$). However, θ_d, as we will show, also depends on Δ and $|c|^2$. Thus there are two quantities to be determined from boundary conditions. This is in contrast to the TG case, for which only one quantity, $|c|^2$, is to be determined from boundary conditions. Knowing Δ and $|c|^2$ is equivalent to knowing the missing boundary conditions at the two faces of the crystal.

We assume $C_3 = 0$ (the PC condition) to be in force. This implies that $c = C_2A_{1d}$ and $d_1 = |C_1|^2 - I_{30} = |c|^2/|C_2|^2$, and using Eq. (4b) one finds an equation connecting Δ and $|c|^2$:

$$\left(|c|^2 - |C_1C_2|^2\right)|\Delta T - \delta|^2 + 4|c|^2|C_2C_4|^2|T|^2 = 0, \tag{5a}$$

where $T = \tanh(\gamma\theta_d)$. On the other hand, using an expression for $c^* = C_1^*A_{20}^* - C_4^*A_{30}^*$ and Eq. (4a), one obtains a quadratic equation for T. The imaginary part of that equation leads to another relation between Δ and $|c|^2$:

$$\left[(|c|^2 + |C_1C_2|^2)\Delta^2 - 4|c|^2|C_2C_4|^2 + 2\Delta|c|^2\epsilon\right] \times T_r - |c|^2(\epsilon + \Delta)\delta = 0, \tag{5b}$$

where $\epsilon = |C_1|^2 + |C_4|^2 - |C_2|^2$ and T_r stands for the real part of T. Finally, from the equation for I^2:

$$\mathrm{d}I^2/\mathrm{d}z = 8|Q_d|^2\gamma_r\exp[2\gamma_r(z - d)], \tag{6}$$

one obtains an expression relating θ_d to Δ and $|c|^2$:

$$\exp(4\gamma_r\theta_d) = \frac{[\exp(-2\gamma_r d) + w^2]^{1/2} + w}{[\exp(-2\gamma_r d) + w^2]^{1/2} - w}\,\frac{(1 + w^2)^{1/2} - w}{(1 + w^2)^{1/2} + w}, \tag{7}$$

where γ_r denotes the real part of γ, $2w = \delta/(|C_2|^4 - |C_2|^2\Delta - |c|^2)^{1/2}$, and θ_0 is assumed to be 0.

Thus one solves Eqs. (5a) and (5b) for Δ and $|c|^2$ and then evaluates θ_d, E, and D:

$$E^2 = \exp(2\gamma\theta_d)\frac{\Delta - \delta}{\Delta + \delta},$$

$$D^2 = \exp(2\gamma\theta_d)\frac{2|c|^2 + |C_2|^2(\Delta + \delta)}{2|c|^2 + |C_2|^2(\Delta - \delta)}. \tag{8}$$

In this manner the solution given by Eqs. (4) is determined. Also, the missing intensities at $z = d$ (or at $z = 0$) are found:

$$I_{1d} = \frac{|c|^2}{|C_2|^2}, \qquad I_{4d} = |C_2|^2 - \Delta - \frac{|c|^2}{|C_2|^2}. \tag{9}$$

One then proceeds to evaluate the four intensities and phases as functions of θ (as in Ref. 5) and, finally, to solve Eq. (3) for $\theta(z)$.

One can find the experimentally interesting quantities without having to go through all the details. For example, the PC reflectivity $\mathcal{R}$ and the diffraction efficiency $\mathcal{D}$ can be found once $|c|^2$ and Δ are known:

$$\mathcal{R} = \frac{I_{30}}{|C_4|^2} = \frac{4|c|^2|T|^2}{|\Delta T - \delta|^2}, \qquad \mathcal{D} = \frac{I_{30}}{|C_2|^2} = \frac{|C_4|^2}{|C_2|^2}\mathcal{R}. \tag{10}$$

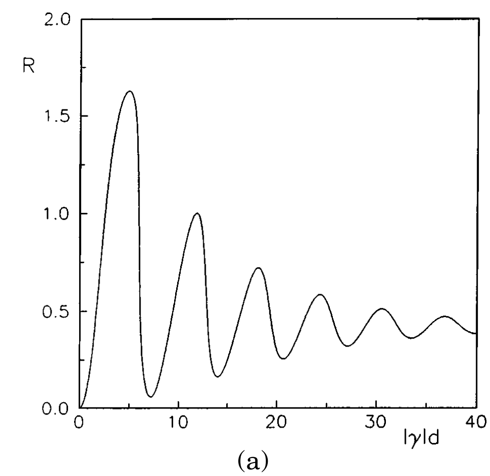

Fig. 1. Reflectivity as a function of the coupling strength, for different arguments of the coupling constant: (a) $\arg(\gamma) = 85°$, (b) $\arg(\gamma) = 89.9°$. The other parameters are $I_{10} = I_{2d} = 1$ and $I_{40} = 0.4$, corresponding to an example considered in Ref. 1.

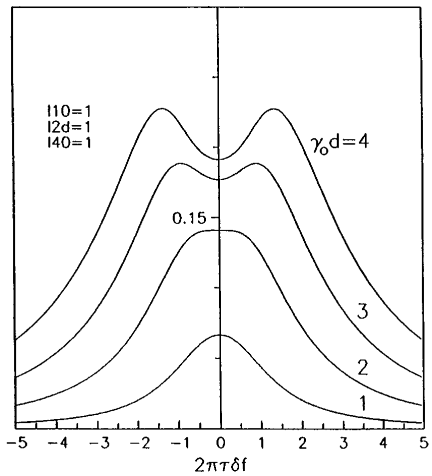

Fig. 2. Diffraction efficiency as a function of the frequency detuning δf. Similar curves appear in TG.

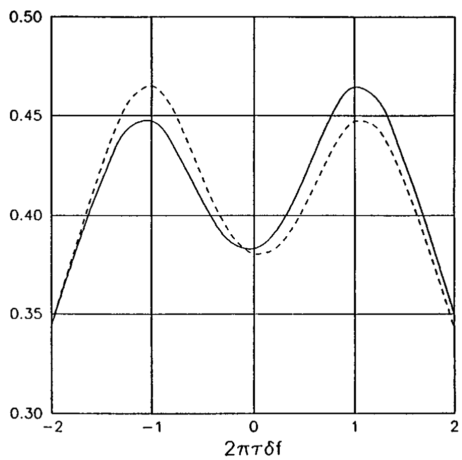

Fig. 3. Diffraction efficiency as a function of the detuning in the transverse case and in TG. The solid curve is for focused Gaussian beams A_{10} and A_{40} and almost flat A_{2d} and A_{3d}; the dashed curve is for focused A_{2d} and A_{3d} and flat A_{10} and A_{40}. Parameters $I_{10} = 1.15$, $I_{2d} = 0.031$, $I_{40} = 1.25$, and $\gamma_0 d = -4.5$ are chosen to correspond to an experiment reported by MacDonald and Feinberg.[6] The PC beam is seeded with $I_{3d} = 0.001$.

This concludes the solution procedure. We present some of our results in Figs. 1–3. For comparison with the known results for the TG,[1] we present in Fig. 1 the reflectivity as a function of the magnitude and the argument of the coupling constant γ. Qualitatively similar results are obtained. As the magnitude increases, one notes the appearance of multistability in the reflectivity.

As an example, we apply the solution to the problem of frequency shifts and detunings in PR oscillators. This is a problem of long history, some controversy, and considerable difficulty on its own.[6,7] As mentioned above, the detuning causes the coupling constant to become complex. Although it is beyond the scope of this Letter to address this problem in depth, we wanted to see whether, as suggested,[6,7] the inclusion of reflection gratings would change the picture of spontaneous detuning. One of the problems in the picture is the appearance of the reflectivity as a function of the detuning. Theory based on TG calculations predicts the existence of two symmetric peaks about zero detuning, whereas experimental results display asymmetry and a lateral shift of the two peaks. Assuming the coupling constant to be of the form

$$\gamma = \frac{\gamma_0}{1 + 2\pi i\tau\delta f}, \tag{11}$$

where τ is the relaxation time constant of the crystal and δf is the frequency detuning, we calculate the diffraction efficiency using Eq. (10). Typical results are depicted in Fig. 2. A two-peak structure appears; however, perfect symmetry and no shift are found. Plane-wave oscillation pictures of PR oscillators, in either TG or RG, cannot resolve the problem of asymmetry and shift in the reflectivity of the PC process.

A qualitatively different picture emerges if one takes into account the transverse spread of beams and allows for (small but necessary) seeding of PC beams. The inclusion of transverse effects breaks the spatial symmetry that plane-wave analysis enforces on the process. Taking Gaussian beams, one can introduce curvature mismatch between different beams and observe changes in the diffraction efficiency. The equations cannot be solved analytically any more. We recently presented a numerical procedure for treating four-wave mixing processes in paraxial approximation.[8] When the procedure is applied to the problem at hand, asymmetry and shift of the two-peak structure in either direction appeared. Typical profiles in the steady state are presented in Fig. 3. In our opinion, any explanation of the problem of spontaneous frequency detuning of PC beams in PR oscillators must include transverse effects.

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