Multigrating phase conjugation: exact results

W. Krolikowski* and M. R. Belić†

Max-Planck-Institut für Quantenoptik D-8046 Garching, Federal Republic of Germany

Received July 22, 1987; accepted October 26, 1987

Multigrating operation of degenerate four-wave mixing in photorefractive media is considered in the strong pump regime. Closed-form expressions for the reflectivity are obtained for the cases of absorption and no absorption. An arbitrary phase mismatch between the intensity interference fringes and the refractive-index gratings is allowed.

The purpose of this Letter is analytically to investigate degenerate four-wave mixing (FWM) in real-time holographic media when more than one refractive-index grating is present. Such a situation is commonly encountered in experiments.[1-3] In fact, single grating operation is achieved only when other grating mechanisms are suppressed by special means (for example, by a choice of polarizations or by application of external electric fields).[1,2] On the other hand, multigrating operation is also envisaged as an efficient mode of operation for high-capacity optical storage elements.[3]

According to the widely accepted theory of the photorefractive effect,[1] there are four contributions to the intensity-induced index gratings in standard FWM geometry: a large-spaced transmission grating, a small-spaced reflection grating, and two additional contributions, usually overlooked, coming from the mixing of the pump beams and of the probe beam and its conjugate. When the pumps are much stronger than the probe and the conjugate, the last contribution can safely be ignored. Then the pump beams are not depleted by the probe or the conjugate beam, and hence their evolution can be solved independently. The four steady-state wave equations in the slowly varying amplitude approximation are of the form

$$IA_1' = -\gamma A_1 A_2^* A_2 - \frac{\alpha}{2} IA_1, \tag{1a}$$

$$IA_2^{*\prime} = -\gamma A_1 A_2^* A_1^* + \frac{\alpha}{2} IA_2^*, \tag{1b}$$

$$IA_3' = \gamma_T(A_1A_4^* + A_2^*A_3)A_2 + \gamma_R(A_1^*A_3 + A_2A_4^*)A_1 + \frac{\alpha}{2}IA_3, \tag{1c}$$

$$IA_4^{*\prime} = \gamma_T(A_1A_4^* + A_2^*A_3)A_1^* + \gamma_R(A_1^*A_3 + A_2A_4^*)A_2 - \frac{\alpha}{2}IA_4^*, \tag{1d}$$

where $I = I_1 + I_2$ is the total intensity, $\gamma = \gamma_r + i\gamma_i$ is the pump mixing constant, α is the intensity absorption coefficient, and γ_T and γ_R are the transmission and the reflection coupling constants, respectively, also complex. Their general form is $in\exp(i\theta)$, where n's are material parameters and θ's are the phase differences between the intensity pattern and the corresponding index grating. Other general assumptions about the geometry, polarizations, etc. are assumed to be the same as in Ref. 1, from which the equations are taken. Our aim is to obtain an expression for the reflectivity $\rho = A_3/A_4^*$ from these equations or, equivalently, to solve the equations. To this end, we apply some of the results on two-wave mixing from Ref. 4 and a procedure for exact treatment of FWM from Ref. 5. The analysis proceeds along the following lines.

Equations (1a) and (1b) describe two-wave mixing of the pumps and are solved exactly in Ref. 4. The solutions are given in the form

$$I_1 = \tfrac{1}{2}\exp(f_0 - \gamma_r z + F),$$
$$I_2 = \tfrac{1}{2}\exp(f_0 - \gamma_r z - F), \tag{2a}$$

where the function F satisfies the following differential equation:

$$F' = \gamma_r \tanh F - \alpha \tag{2b}$$

and f_0 is a constant of little concern to us. The sum phase of the pumps satisfies another equation:

$$\psi' = -\gamma_i \frac{I_2 - I_1}{I_1 + I_2}, \tag{3}$$

which can be solved once the intensities are known. It can also be given in terms of the function F: $\psi = \psi_0 + \gamma_i(F - F_0)/\gamma_r + \alpha\gamma_i z/\gamma_r$. This information is to be used in the equation for ρ, obtained from Eqs. (1c) and (1d):

$$\rho' = \frac{(I_1I_2)^{1/2}}{I}(\gamma_R + \gamma_T)[1 - \exp(-2i\psi)\rho^2]\exp(i\psi) + \frac{I_2 - I_1}{I}(\gamma_T - \gamma_R)\rho + \alpha\rho. \tag{4}$$

Explicit dependence on ψ is eliminated by defining a new dependent variable: $R = \rho\exp(-i\psi)$, whose phase equals the relative phase of the FWM process. Consequently, R satisfies the following simplified equation:

$$R' = \frac{\kappa_1}{I}(I_1I_2)^{1/2}(1 - R^2) + \frac{\kappa_2}{I}(I_2 - I_1)R + \alpha R, \tag{5}$$

0146-9592/88/020149-03$2.00/0

where $\kappa_1 = \gamma_T + \gamma_R$ and $\kappa_2 = \gamma_T - \gamma_R + i\gamma_i$. As a result of the strong-pump limit, the total intensity approximately equals $I_1 + I_2$, and all coefficients in this equation are known functions. Equation (5) is a Riccati equation of the type that we encountered in Ref. 5. We will use a similar type of manipulation to reduce it to the hypergeometric or a related linear differential equation. Of course, the situation is more complicated now, since we deal with multiple gratings. This will necessitate separate treatments of the absorptionless case and of the case with absorption.

When the expressions for I_1 and I_2 are substituted into Eq. (5), it is seen that a more convenient independent variable is the function F. The new equation is of the form

$$(\gamma_r \tanh F - \alpha)R' = \frac{\kappa_1}{2}\frac{1-R^2}{\cosh F} - \kappa_2 \tanh FR + \alpha R, \tag{6}$$

and the differentiation is now with respect to F. In the usual treatment of a Riccati equation, a new dependent variable is introduced: $R \rightarrow [2(\gamma_r \sinh F - \alpha \cosh F)/\kappa_1](v/v')$. Thus a second-order linear differential equation for v is obtained:

$$v'' + \left(\frac{\gamma_r - \alpha \tanh F - \alpha}{\gamma_r \tanh F - \alpha} + \frac{\kappa_2}{\gamma_r - \alpha \coth F}\right)v' - \frac{\kappa_1^{\,2}}{4(\gamma_r \tanh F - \alpha)^2 \cosh^2 F}v = 0. \tag{7}$$

This apparently hopelessly complex equation is actually only one transformation away from the hypergeometric equation. The most convenient form of the transformation, however, depends on whether $\alpha = 0$ or $\alpha \neq 0$.

When $\alpha = 0$, Eq. (7) becomes

$$v'' + \left(\coth F + \frac{\kappa_2}{\gamma_r}\right)v' - \frac{\kappa_1^{\,2}}{4\gamma_r^{\,2}}\operatorname{cosech}^2 Fv = 0, \tag{7$'$}$$

where, for simplicity, the same set of symbols F and v is used to denote the variables. The appropriate transformation of the independent variable is then $2\xi = 1 + \coth F$ if $F > 0$ and $2\xi = 1 - \coth F$ if $F < 0$, and the equation becomes

$$\xi(\xi - 1)v'' + (\xi - c)v' - \frac{\kappa_1^{\,2}}{4\gamma_r^{\,2}}v = 0, \tag{8}$$

where c equals $1/2 - \kappa_2/2\gamma_r$ for $F > 0$ and $1/2 + \kappa_2/2\gamma_r$ for $F < 0$. Equation (8) is the hypergeometric equation, whose properties and fundamental solutions are well known.[6] Note that the choice of the variable ξ is such that $\xi \geq 1$, so that only the solutions around the regular singular points 1 and ∞ are needed. The reflectivity is finally given by

$$\rho_0 = \frac{2\gamma_r \sinh F_0 \exp(i\psi_0)}{\kappa_1}\frac{v_{1d}'v_{20}' - v_{2d}v_{10}'}{v_{1d}'v_{20} - v_{2d}'v_{10}}, \tag{9}$$

where v_1 and v_2 are the fundamental solutions of Eq. (8), to be found, for example, in Kummer's list.[6]

Note that in this formulation the split-boundary-value problem is simplified, i.e., transferred to the two-wave mixing problem. That is, Eq. (8) does not contain unknown coefficients, and so v_{10}, v_{1d}, etc. are explicitly known. The only unknown quantity in ρ is F_0 (ψ_0 can be made zero), and the procedure for finding F_0 is outlined in Ref. 4.

For α nonzero, the more convenient form of the variable change is $2\xi = 1 + \tanh F$. Equation (7) then becomes

$$\xi(\xi - 1)v'' - \left(1 - 2\xi + \frac{\xi\dfrac{\kappa_2 - \alpha}{2\gamma_r} + \dfrac{\gamma_r + \kappa_2}{4\gamma_r}}{\xi - \dfrac{\gamma_r + \alpha}{2\gamma_r}}\right)v' + \frac{\kappa_1^{\,2}}{16\gamma_r^{\,2}\left(\xi - \dfrac{\gamma_r + \alpha}{2\gamma_r}\right)^2}v = 0, \tag{10}$$

which is a special case of the Papperitz (or Riemann) equation.[6] The Papperitz equation is the most general second-order linear differential equation with three regular singularities. The hypergeometric equation is only a special case of the Papperitz, with the singularities located at 0, 1, and ∞. However, since the regular singular points are movable, our equation can still be transformed into the hypergeometric equation. This will be accomplished by yet another transformation of the independent variable:

$$\eta = \frac{\xi\dfrac{\gamma_r - \alpha}{2\gamma_r}}{\xi - \dfrac{\gamma_r + \alpha}{2\gamma_r}}, \tag{11}$$

and the final equation for the absorption case is

$$\eta(1 - \eta)v'' + [c - \eta(a + b + 1)v' - abv = 0, \tag{12}$$

with

$$c = \frac{2\alpha + \gamma_r - \kappa_2}{2(\gamma_r + \alpha)}, \tag{13a}$$

$$a + b = \frac{\alpha(\gamma_r - \kappa_2) - 2\gamma_r\kappa_2}{\gamma_r^{\,2} - \alpha^2}, \qquad ab = \frac{\kappa_1^{\,2}}{4(\gamma_r^{\,2} - \alpha^2)}. \tag{13b}$$

Reflectivity is then evaluated as in the absorptionless case. From these expressions a condition for the self-oscillation follows in a general form:

$$v_{1d}'v_{20} - v_{2d}'v_{10} = 0, \tag{14}$$

provided that the zeros exist. Here, naturally the form and the meaning of independent and dependent variables are different for the absorption and the absorptionless cases.

In Fig. 1 the effect of absorption and pump coupling on the intensity reflectivity $|\rho_0|^2$ is displayed. While the effect of absorption is as expected, the pump cou-

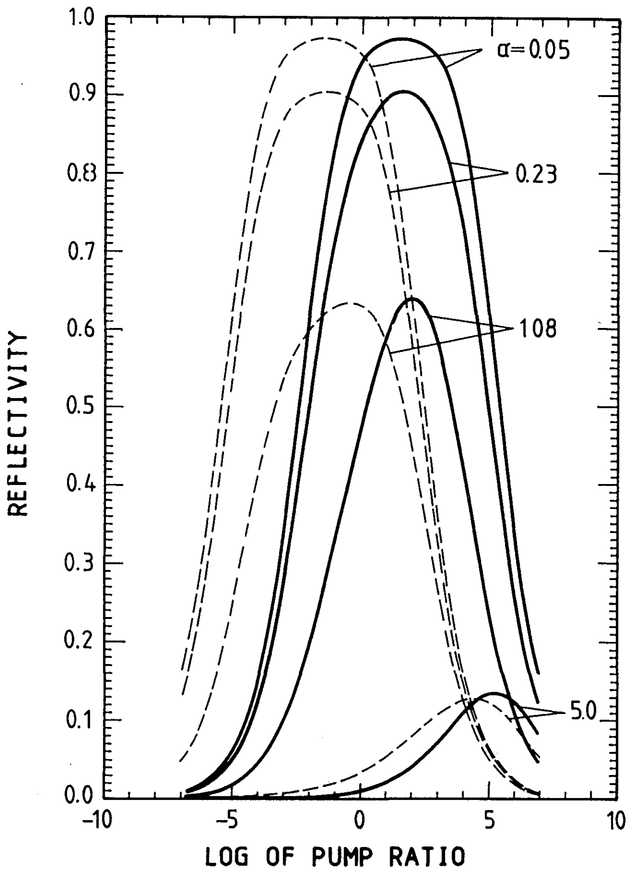

Fig. 1. Intensity reflectivity $|\rho_0|^2$ as a function of the pump ratio I_{2d}/I_{10} and for different absorptions (given in reciprocal centimeters). Transmission and reflection couplings are $\gamma_T = \gamma_R = 10\ \mathrm{cm}^{-1}$, and the pump coupling varies from $-5\ \mathrm{cm}^{-1}$ (solid lines) to $+5\ \mathrm{cm}^{-1}$ (dashed lines). The thickness of the crystal is set to $d = 0.3$ cm.

pling causes a horizontal shift of the reflectivity curves along the pump ratio axis, a consequence of the energy transfer between the pumps. The figure also displays saturation of the reflectivity at $|\rho_0|^2 = 1$, another characteristic feature of equal-strength multigrating phase conjugation.

A legitimate question to be posed is what happens in the case when $\gamma_r = 0$. In the experiment the pump mixing and especially the probe–conjugate mixing are not pronounced,[7] and in some media (and in most of the theories) this mixing is not even present. In this case Eq. (7) becomes

$$v'' + \left[\left(1 - \frac{\kappa_2}{\alpha}\right)\tanh F + 1\right]v' - \frac{{\kappa_1}^2}{4\alpha^2}\,\mathrm{sech}^2 F v = 0, \tag{7''}$$

with F being $F = F_0 - \alpha z$. This is exactly the equation considered in detail in Ref. 5, and there is no need to repeat the analysis here.

In conclusion, we have considered multigrating degenerate FWM in real-time holographic media. Closed-form expressions for the phase-conjugate reflectivity are obtained for the absorptionless case and for the case when absorption is taken into account. An arbitrary phase mismatch between the interference fringes and the index gratings is allowed.

The authors would like to thank H. Walther for hospitality at the Max-Planck-Institut für Quantenoptik. M. R. Belić is also grateful to the Alexander von Humboldt Foundation for financial support.

* Permanent address, Institute of Physics, Polish Academy of Sciences, 02-668 Warsaw, Poland.

† Permanent address, Institute of Physics, P.O. Box 57, 11001 Belgrade, Yugoslavia.

References

1. M. Cronin-Golomb, B. Fisher, J. O. White, and A. Yariv, IEEE J. Quantum Electron. **QE-20,** 12 (1984).
2. R. A. Fisher, ed., *Optical Phase Conjugation* (Academic, New York, 1983); Y. H. Ja, Opt. Commun. **41,** 159 (1982).
3. P. Günter, Phys. Rep. **93,** 199 (1982); P. Hariharan, *Optical Holography* (Cambridge U. Press, Cambridge, 1984).
4. M. R. Belić, Opt. Quantum Electron. **16,** 551 (1984).
5. M. R. Belić, Opt. Lett. **12,** 105 (1987).
6. M. Abramowitz and J. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1953); P. M. Morse and H. Feshbach, *Mathematical Methods of Theoretical Physics* (McGraw-Hill, New York, 1953).
7. D. J. Gauthier, P. Narum, and R. W. Boyd, Phys. Rev. Lett. **58,** 1640 (1987).