EXACT SOLUTION TO THE STATIONARY HOLOGRAPHIC FOUR-WAVE MIXING IN PHOTOREFRACTIVE CRYSTALS ☆

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An exact solution to the stationary holographic degenerate four-wave mixing in both transmission and reflection geometry for photorefractive media is obtained. The effects of pump depletion and light absorption in the non-linear dynamic medium are rigorously taken into account. The case of the transmission geometry is treated in more detail, with an emphasis on the energy transfer. The numerical steps in the procedure are reduced to the solution of one simple first-order differential equation, or are absent altogether for the case of no absorption.

Due to great applicability in various branches of nonlinear optics, a substantial amount of work, both experimental [1–5] and theoretical [4–8] went into degenerate four-wave mixing (DFWM) in recent years. On the theoretical side, progress made in selection of the effective wave-coupling media placed a more urgent need for solutions of wave-mixing theories in which pump depletion and light absorption in the dynamic medium are allowed for. While absorption can be neglected in some weakly absorbing crystals, for example in lithium niobite, in other popular crystals such as BSO it can be as high as 10 cm^{-1}, and therefore must be included into any realistic theory. In this report we present a method for solution of the coupled wave equations describing energy transfer in the stationary holographic DFWM in both transmission and reflection geometry, with allowance for depletion and absorption in the medium. The emphasis is placed on the case of transmission geometry, and the results on reflection geometry are only mentioned for completeness. A more complete analysis of the reflection geometry has been published elsewhere [9].

We consider here the standard beam configuration for DFWM. Two pump beams A_1 and A_2 impinge from the opposite sides on a photorefractive crystal situated in between the planes $z = 0$ and $z = d$. For a certain nonlinear coupling of waves in the crystal [7], a signal beam A_3 (incident say from the side of A_1) causes generation of the counter-propagating phase-conjugated beam A_4. In a transmission geometry the prevalent grating is formed between the waves A_3 and A_1, while a portion of the wave A_2 gets diffracted (transmitted) into the phase-conjugated reconstruction A_4. In a reflection geometry gratings are formed predominantly between the waves A_3 and A_2, and the wave A_1 is reflected off the grating into the conjugated wave A_4. Variation of the waves in each geometry is described by a corresponding set of nonlinear coupled-wave equations.

For the case of the transmission grating the set of

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wave equations in a slowly varying amplitude approximation is of the form [7]:

$$dI_1/dz = -\alpha I_1 + g[I_1I_3 + (I_1I_2I_3I_4)^{1/2}]/I_0, \quad (1a)$$

$$dI_2/dz = \alpha I_2 + g[I_2I_4 + (I_1I_2I_3I_4)^{1/2}]/I_0, \quad (1b)$$

$$dI_3/dz = -\alpha I_3 - g[I_1I_3 + (I_1I_2I_3I_4)^{1/2}]/I_0, \quad (1c)$$

$$dI_4/dz = \alpha I_4 - g[I_4I_2 + (I_1I_2I_3I_4)^{1/2}]/I_0, \quad (1d)$$

where I_1, I_2, I_3, I_4 represent the beam intensities, $I_0 = I_1 + I_2 + I_3 + I_4$ is the total intensity, α is the linear absorption coefficient, and g the effective coupling constant. For the case of the reflection geometry we similarly have [8]:

$$dI_1/dz = -\alpha I_1 - 2g[I_1I_4 + (I_1I_2I_3I_4)^{1/2}]/I_0, \quad (2a)$$

$$dI_2/dz = \alpha I_2 - 2g[I_2I_3 + (I_1I_2I_3I_4)^{1/2}]/I_0, \quad (2b)$$

$$dI_3/dz = -\alpha I_3 - 2g[I_3I_2 + (I_1I_2I_3I_4)^{1/2}]/I_0, \quad (2c)$$

$$dI_4/dz = \alpha I_4 - 2g[I_4I_1 + (I_1I_2I_3I_4)^{1/2}]/I_0. \quad (2d)$$

For simplicity the same set of symbols α, g (assumed to be real) is used to denote material parameters in both cases. Without loss of generality, we consider the case $g > 0$. A similar analysis carries through for the case $g < 0$. Note that we are concerned with the stationary energy transfer and the generation of the phase-conjugated wave, so that only the fundamental components of the phase grating are taken into account. Also, a $\pi/2$ shift between the interference fringes and the index grating is introduced, so that the theory in this form applies to the experimentally interesting case of photorefractive media. The boundary conditions are applied at two end-points, that is $I_1(0) = 1, I_2(d) = C_2, I_3(0) = C_3$, and $I_4(d) = 0$ is assumed to be known.

The physics of DFWM and real-time holography has been considered at length elsewhere [1,4,5,7] and will not be of major concern here. The main result of this report is a procedure for exact solution of the system of equations (1) and (2) with or without absorption. Again, the case of the transmission geometry will be treated in more details, and the results on the reflection geometry will be only cited for completeness. Along the way a comparison will be made with the results of Cronin-Golomb et al. [7] who treated (by a different method) transmission and reflection geometry without absorption, and with Ja [8], who treated the complete reflection geometry numerically.

Analysis of eqs. (1) proceeds as follows. Firstly new dependent variables are introduced: $u_1 = I_1 + I_3$, $u_2 = I_2 + I_4, v_1 = I_1 - I_3, v_2 = I_2 - I_4$. In terms of these variables and two auxiliary functions $f_1^2 = u_1^2 - v_1^2, f_2^2 = u_2^2 - v_2^2$, eqs. (1) transform into:

$$u_1' = -\beta u_1, \quad u_2' = \beta u_2, \quad (3a)$$

$$v_1' = -\beta v_1 + \tfrac{1}{2}f_1(f_1 + f_2)/(u_1 + u_2), \quad (3b)$$

$$v_2' = \beta v_2 + \tfrac{1}{2}f_2(f_1 + f_2)/(u_1 + u_2), \quad (3c)$$

where the prime denotes the derivative with respect to gz, and $\beta = \alpha/g$. Therefore $u_1 = (1 + C_3)\exp(-\alpha z)$, $u_2 = C_2 \exp \alpha(z - d)$, and $v_1v_2 + f_1f_2 = \text{const} = (I_{1d} - I_{3d})C_2$, so that only one of the equations (either (3b) or (3c)) remains to be solved. Assuming a solution of the form $f_1 = u_1 \sin x, v_1 = u_1 \cos x$, $f_2 = u_2 \sin y, v_2 = u_2 \cos y$, it follows: $x = y + \cos^{-1}(a/b)$, where $a = v_{1d}v_{2d} = (I_{1d} - I_{3d})C_2$ and $b = u_{1d}u_{2d} = (1 + C_3)C_2 \exp(-\alpha d)$. Also eq. (3b) or (3c) becomes:

$$-2y' = [(u_2^2 + a)/(u_2^2 + b)] \sin y + [c/u_2^2 + b)] \cos y, \quad (4)$$

with $c^2 = b^2 - a^2$. The intensities of the four waves are given by:

$$I_1 = u_1 \cos^2[(y + Y)/2], \quad I_2 = u_2 \cos^2(y/2), \quad (5a)$$

$$I_3 = u_1 \sin^2[(y + Y)/2], \quad I_4 = u_2 \sin^2(y/2), \quad (5b)$$

where $Y = \cos^{-1}(a/b)$. Eq. (4), and its Riccati equivalent (brought about by a variable change $w = \tan y$), as well as its linear second-order counterpart, seem to be amenable to analytic evaluation only in the case $\alpha = 0$. The solution is then of the form:

$$\tan(y/2) = \frac{\{\exp[\mu(d - z)] - 1\} \tan(\theta/2)}{1 + \exp[\mu(d - z)] \tan^2(\theta/2)}. \quad (6)$$

Here $\tan\theta = c/(C_2^2 + a)$, and $\mu = g[(1 + C_3)^2 + 2a + C_2^2]^{1/2}/2(1 + C_2 + C_3)$. In the general case $\alpha \neq 0$ solution has to be performed numerically, starting say at $z = d$ (with an initial value $y_d = 0$) and going backwards to $z = 0$.

To complete the solution, it remains only to determine v_{1d} or a (which figure explicitly in the expressions for intensity through y and Y) in terms of the given boundary values. For $\alpha = 0$ this is accomplished by solving an implicit algebraic equation for a:

$$\frac{c\sqrt{C_3} - b + a}{c + \sqrt{C_3}(b - a)} = \frac{[\exp(\mu d) - 1]\tan(\theta/2)}{1 + \exp(\mu d)\tan^2(\theta/2)}. \tag{7}$$

For $\alpha \neq 0$ the value of the parameter v_{1d} is found as the root of the equation:

$$y_0(v_{1d}) + \cos^{-1}\left(\frac{v_{1d}\exp(\alpha d)}{1 + C_3}\right) = 2\tan^{-1}(\sqrt{C_3}), \tag{8}$$

where the function $y_0(v_{1d})$ is known from the numerical integration.

We note agreement between our results and that of Cronin-Golomb et al. [7] for the case $\alpha = 0$ once the following identification is made: their $2|c|^2$ equal $b + a$, $\Delta + 4|c|^2$ equal $u_1^2 + 2a + u_2^2$, etc. In our notation the intensity reflectivity is $R = (C_2/C_3)\sin^2(y_0/2)$.

By way of an example, let us demonstrate how the procedure works for a specific set of parameters and boundary values. To the rather simple numerical problem at hand we apply elementary methods: an initial-value integration of eq. (4), and a graphical solution of nonlinear algebraic equations (7) and (8). All the numerical steps in this report are performed on a TI 59 handheld calculator. We pick $C_2 = 1, C_3 = 0.7$ (in units of $I_1(0)$), $g = 5\ \text{cm}^{-1}$, $d = 0.2$ cm, and consider three values for the absorption constant α: 0, 3 cm^{-1}, and 8 cm^{-1}. In fig. 1 the graphical solution of eqs. (7) and (8) (i.e. the evaluation of a) is depicted. The solution is estimated graphically, and then iterated for greater accuracy. Using the found values for a, in fig. 2 the corresponding functions $y(z)$ are plotted.

We note that y is rather small for realistic values of the parameters, so that eq. (4) can be linearized with little error. The solution of the linearized eq. (4) is easily found:

$$y = \tfrac{1}{2}g\exp[G(z)]\int_z^d \frac{c}{u_2^2 + b}\exp[-G(\zeta)]\,d\zeta, \tag{9a}$$

where:

$$G(z) = (ga/2b)(z - d) - (g/4\alpha b)(b - a)\ln\left(\frac{C_2^2 + b}{u_2^2 + b}\right). \tag{9b}$$

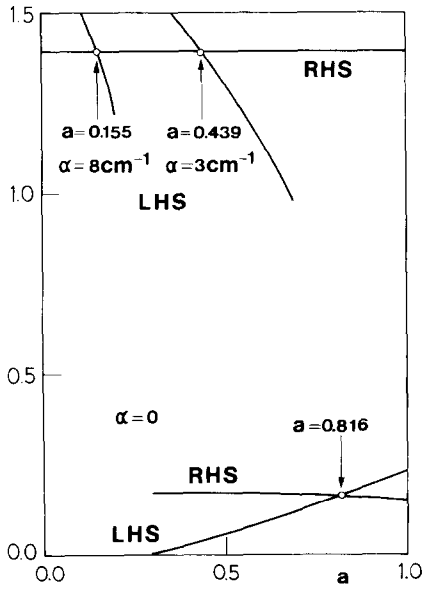

Fig. 1. Graphical solution of the algebraic equations (7) and (8) for the following set of parameters: $C_2 = 1$, $C_3 = 0.7$, $g = 5\ \text{cm}^{-1}$, $d = 0.2$ cm. RHS and LHS stand for the right-hand side and the left-hand side of the aforementioned equations. The solutions appear to be unique for realistic values of the parameters. Note that $v_{1d} = a$ for the chosen set of boundary values.

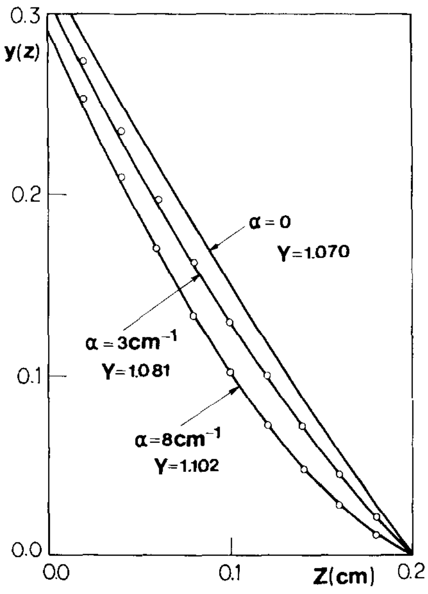

Fig. 2. Function $y(z)$ needed in specification of the intensities, for three different values of the absorption coefficient. The points indicated atop the curves for $\alpha \neq 0$ correspond to the exact solution of the linearized eq. (4).

A set of points calculated using the linearized solution is also plotted in fig. 2. Naturally, the above qualitative estimates worsen as the value of g (or rather gd) increases. But, by the same token, for any $\alpha \neq 0$, as g increases, the corresponding $y(z)$ is getting closer to the solution for $\alpha = 0$. In this manner the whole procedure can be made analytic, and reduced to the manipulation of algebraic equations.

In fig. 3 the corresponding intensities are presented. It is seen that the effect of absorption is always deleterious – it attenuates the beams regardless of the geometry. By an inspection of fig. 2 it is also seen that the cosine and the sine terms in the intensities (especially in I_1 and I_3) vary little with the absorption, so that the major change in the intensities comes from the factor $\exp(\pm\alpha z)$ situated in u_2 and u_1 respectively. This validates, for example the conjecture of Yeh [10], who noticed that his theory of two-wave mixing in photorefractive media can be corrected for absorption to a good degree by the use of the factor $\exp(\pm\alpha z)$. The absorption also lowers the reflectivity roughly by a factor $\exp(-\alpha d)$, the expression for the reflection coefficient being now:

$$R = (C_2/C_3)\exp(-\alpha d)\sin^2(y_0/2). \tag{10}$$

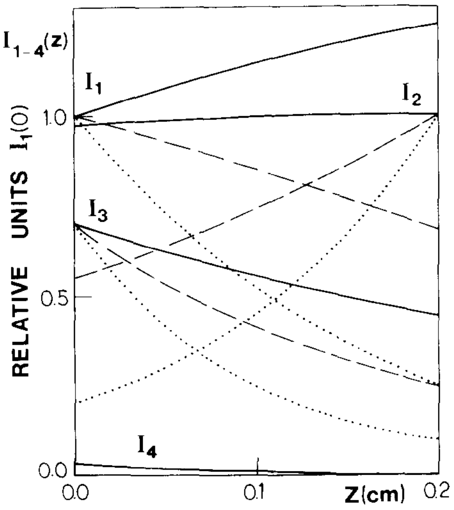

Fig. 3. Intensities of the four beams inside the crystal for three values of the absorption: no absorption (the full lines), 3 cm^{-1} (dashed lines), and 8 cm^{-1} (dotted lines). The attenuated (dashed and dotted) lines for the phase-conjugated beam I_4 are not shown.

We note that y_0 is an increasing function of the thickness of the medium d, starting from $y_0 = 0$ at $d = 0$. Since the oscillatory sine-term in this expression is multiplied by an $\exp(-\alpha d)$ factor, there appears to exist for each $\alpha \neq 0$ an optimal thickness of the hologram that maximizes the value of the reflectivity.

The presented procedure for treatment of the energy transfer in the holographic DFWM can be applied to the problem of the phase transfer as well, with less analytical results along the way, and more of the necessary numerics. Nonetheless, the important case of the small phase transfer can still be treated exactly, with the relative phase $\phi = \phi_3 + \phi_4 - \phi_1 - \phi_2$ given by:

$$\ln(\phi/\phi_d) = -\tfrac{1}{2}g \int_z^d [(u_2/I_0)\cos y \tan(y + Y) + (u_1/I_0)\cos(y + Y)\tan y]. \tag{11}$$

Analysis of the reflection geometry proceeds along similar lines. The general solution, as it appears in ref. [9] is of the form:

$$I_1 = \tfrac{1}{2}\exp(\psi - u)\cos^2[(p + q)/2], \tag{12a}$$

$$I_2 = \tfrac{1}{2}\exp(\psi + u)\cos^2[(p - q)/2], \tag{12b}$$

$$I_3 = \tfrac{1}{2}\exp(\psi - u)\sin^2[(p + q)/2], \tag{12c}$$

$$I_4 = \tfrac{1}{2}\exp(\psi + u)\sin^2[(p - q)/2], \tag{12d}$$

where four z-dependent functions ψ, p, u, q are determined in the following way. The first two of the four are connected with the integration constants by $\exp(\psi)\cos p = E = 2\sqrt{I_{1d}C_2}$ and $\exp(\psi)\sin p = F = F_d \exp g(d - z)$; u is the solution of a simple first-order differential equation:

$$u' = \sin^2 p \tanh u + \beta, \tag{13}$$

and q is a simple functional of u:

$$q = q_d - (E/F)v + (E/F_d)v_d - (gE/F_d)\exp(-gd) \int_z^d v \exp(g\zeta)d\zeta, \tag{14}$$

where $v = u - \alpha z$. Thus the solution of eqs. (2) is reduced to the integration of a simple differential equation, eq. (13); a quadrature, eq. (14); and a knowledge of the two constants of integration, E and F_d. The

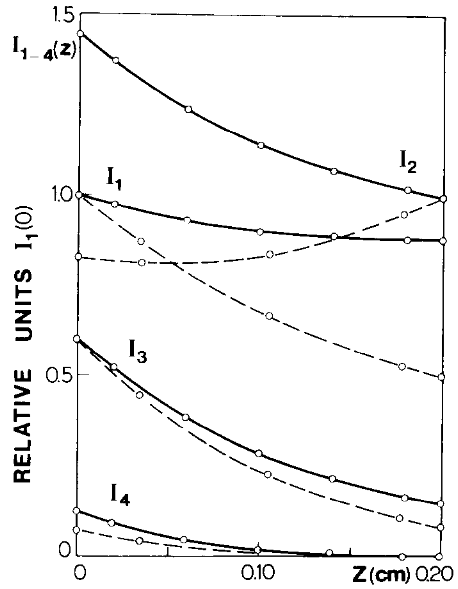

Fig. 4. Comparison of the exact solution for reflection grating with the numerical results of ref. [8]. The curves are taken from fig. 2 in ref. [8] (the full lines correspond to $\alpha = 0$, the dashed to $\alpha = 3\ \mathrm{cm}^{-1}$), and the points indicated are obtained from the exact solution, eqs. (12). The apparently much higher level of phase conjugation here as compared to the transmission grating in fig. 3 is caused by the much stronger beam coupling constant (note g in eqs. (1), and $2g$ in eqs. (2)).

constants of integration, together with the initial values u_d and q_d needed for specification of the intensities are found by applying boundary conditions to the solution given by eqs. (12). In fig. 4 the comparison with the numerical results of ref. [8] is depicted, for the values of the parameters: $C_2 = 1$, $C_3 = 0.6$, $g = 6\ \mathrm{cm}^{-1}$, $d = 0.2$ cm, and for the two values of α: 0 and 3 cm^{-1}. As expected, no noticeable difference is evident.

In conclusion, we propose a procedure for exact solution of the stationary holographic DFWM in photorefractive crystals, in both transmission and reflection geometry when the absorption in the dynamic medium is taken into account, and depletion of the pumps allowed for.

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