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Invited Paper

Chirped femtosecond pulses in the higher-order nonlinear Schrödinger equation with non-Kerr nonlinear terms and cubic–quintic–septic nonlinearities

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ABSTRACT

We consider a high-order nonlinear Schrödinger equation with competing cubic–quintic–septic nonlinearities, non-Kerr quintic nonlinearity, self-steepening, and self-frequency shift. The model describes the propagation of ultrashort (femtosecond) optical pulses in highly nonlinear optical fibers. A new ansatz is adopted to obtain nonlinear chirp associated with the propagating femtosecond soliton pulses. It is shown that the resultant elliptic equation of the problem is of high order, contains several new terms and is more general than the earlier reported results, thus providing a systematic way to find exact chirped soliton solutions of the septic model. Novel soliton solutions, including chirped bright, dark, kink and fractional-transform soliton solutions are obtained for special choices of parameters. Furthermore, we present the parameter domains in which these optical solitons exist. The nonlinear chirp associated with each of the solitonic solutions is also determined. It is shown that the chirping is proportional to the intensity of the wave and depends on higher-order nonlinearities. Of special interest is the soliton solution of the bright and dark type, determined for the general case when all coefficients in the equation have nonzero values. These results can be useful for possible chirped-soliton-based applications of highly nonlinear optical fiber systems.

1. Introduction

Propagation of soliton pulses in nonlinear media has drawn considerable attention in a range of physical settings, such as plasma physics [1,2], fluid dynamics [3], fiber-optic communications and photonics [4,5], Bose–Einstein condensates [6–8], and nuclear physics [9]. Two different types of envelope solitons, bright and dark, can propagate in nonlinear dispersive media [10]. The unique property of optical solitons, either bright or dark, is their particle-like behavior in interaction [11].

In nonlinear optics, the nonlinear Schrödinger (NLS) equation is a generic model for describing the dynamics of light pulses in Kerr nonlinear media. For picosecond light pulses, the NLS equation includes only the group velocity dispersion (GVD) and the self-phase modulation, well known in fibers, and it admits bright and dark soliton-type pulse propagation in anomalous and normal dispersion regimes, respectively [12]. The possibility of bright (dark) solitons in optical fibers is due to the exact counterbalancing between the effects of anomalous (normal) group velocity dispersion and self-phase modulation [13]. These optical structures have been studied extensively, both experimentally and theoretically [14–21].

However, modeling the propagation of subpicosecond optical pulses in highly nonlinear fibers requires the use of a higher-order nonlinear Schrödinger (HNLS) equation. For example, at high light intensities, the NLS equation should also include the $\chi^{(5)}$ nonlinearity (cubic–quintic medium) [22]. In general a self-defocusing fifth-order susceptibility $\chi^{(5)}$ is needed to account for the saturation of $\chi^{(3)}$ [23]. The nonlinearity arising due to fifth order susceptibility can be obtained in many optical materials, such as semiconductors, semiconductor doped glasses, polydiacctylene toluene sulfonate (PTS), calcogenide glasses, and some transparent organic materials [24]. However, when the saturation is very strong, a self-focusing $\chi^{(7)}$ is also needed [23]. It is interesting to note that the experimental measurements of fifth- and seventh-order nonlinearities of several glasses using the spectrally resolved

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two-beam coupling technique has been reported in Ref. [25]. The competition between the three types of cubic, quintic and septic nonlinearity in a material can drastically modify its behavior and the dynamic properties and existence of soliton pulses. In recent years attention has turned to the investigation of chirped solitons propagation in nonlinear dynamical systems. Applications of such objects include pulse compression or amplification, optical pulse compressors, and solitary-wave-based communication links [26,27].

Finding exact chirped solitonlike solutions to NLS-type models is one of the essential tasks of nonlinear optics. Such localized structures offer a rich platform for studying propagation properties of optical pulses in various nonlinear materials. However, much of it is confined to Kerr nonlinearity in the equations and pulses with linear chirp. Hmurcik and Kaup [28] have investigated the pulse with linear chirp and a hyperbolic-secant amplitude profile numerically. In the context of a nonlinear chirp, the authors of Refs. [29,30] solved the NLS equation with self-steepening and self-frequency shift effects and obtained solitonlike solutions with nonlinear chirp. Recently, Alka et al. [31] found chirped soliton solutions with quintic nonlinearity term added to the NLS equation with self-steepening and self-frequency shift. Bright and dark traveling and solitary chirped waves in a three-dimensional medium with all three nonlinearities present – cubic, quintic and septic – have been found in [32].

The objective of this paper is to investigate the chirped soliton propagation for the HNLS equation under the influence of the cubic–quintic–septic and non-Kerr quintic nonlinearities. Investigation of femtosecond pulse propagation in media with higher-order Kerr nonlinear response, which is highly intensity dependent, is needed to understand diverse nonlinear phenomena arising in some applications (e.g., ultrafast optical switching). Here, we derive families of chirped solitonlike solutions for the HNLS equation, by adopting a nonlinear chirping ansatz which differs from that used in Ref. [31]. We further derive the corresponding chirp associated with each of these families. Furthermore, we present the parameter domains in which the chirped solutions exist.

The paper is organized as follows: the model of equation with cubic–quintic–septic nonlinearities, which describes the envelope soliton propagation in non-Kerr media with high-order nonlinearities is presented in Section 2. The nonlinear chirp *ansatz* that is used to determine the chirping associated with propagating femtosecond soliton pulses is also introduced. In Section 3, families of chirped solitonlike solutions have been found under certain parametric conditions. The corresponding chirp associated with each of these families is also determined. The exact chirped soliton solution for the considered model is also determined in the general case where there are no constraints on the number of parameters. Finally, we summarize our findings in Section 4.

2. Governing equation

The HNLS equation with non-Kerr nonlinear terms and cubic–quintic–septic nonlinearities, modeling the propagation of an ultrashort femtosecond optical pulses can be written in the form

$$iE_z + a_1E_{tt} + a_2|E|^2E + a_3|E|^4E + a_4|E|^6E + ia_5\left(|E|^2E\right)_t + ia_6E\left(|E|^2\right)_t + ia_7\left(|E|^4E\right)_t + ia_8E\left(|E|^4\right)_t = 0, \tag{1}$$

where $E(z, t)$ is the complex envelope of the electric field, a_1, a_5 and a_6 are parameters related to group velocity dispersion (GVD), self-steepening, and self-frequency shift, respectively. The terms related to the coefficients a_2, a_3 and a_4 represent the cubic, quintic and septic nonlinearities of the medium, respectively. The parameter a_7 and a_8 account for the quintic non-Kerr nonlinearity terms.

Model (1) combining the cubic, quintic and septic nonlinearities can be used, for instance, in the description of the propagation of ultrashort optical pulses in highly nonlinear media. Unlike the NLS equation, this model with higher-order nonlinear terms is not integrable in general. Obviously, when the last six terms of Eq. (1) can be neglected, the equation reduces to the standard NLS equation, which describes the propagation of picosecond pulses in standard optical fibers, applicable for instance in the context of optical communications. Recently, the model in Eq. (1) without the septic nonlinear term [i.e., $a_4 = 0$] but including the third-order and fourth-order dispersions was used to investigate the modulation instability (MI) phenomenon in optical fibers [24], and in searching for multipole solitary wave solutions [33]. In Ref. [23], the effects of septic nonlinearity on the MI phenomenon were studied within the framework of Eq. (1) when the third-order and fourth-order dispersions are considered. Furthermore, the dark-in-the-bright solution, also called the dipole soliton of the HNLS equation (1), including the third-order dispersion, has been presented recently in the limit $a_4 = 0$ [34]. In addition, wave solutions for the NLS equation with polynomial and triple power law nonlinearities have been obtained in [32,35–38].

Notice that the model (1) constitutes an extension of the equation adopted in Ref. [31], where solitonlike dark (bright) solitons and double-kink solitons solutions with nonlinear chirp have been obtained under certain parametric conditions. In particular, the non-Kerr nonlinear terms, $ia_7(|E|^4E)_t$ and $ia_8E(|E|^4)_t$, which did not appear in Ref. [31], emerge along with the septic nonlinearity term $a_4|E|^6E$. Here, for the first time to our knowledge, we consider the effect of septic nonlinearity and quintic non-Kerr terms and obtain chirped soliton solutions with a different form of nonlinear chirping.

We start from the representation of the complex field in the form [31]:

$$E(z, t) = \rho(\xi)e^{i[\chi(\xi)-kz]}, \tag{2}$$

where ρ and χ are real functions of the traveling coordinate $\xi = t - uz$. Here u is given in terms of the group velocity of the wave packet as $u = 1/v$. The corresponding chirp is given by $\delta\omega(t, z) = -\frac{\partial}{\partial t}[\chi(\xi) - kz] = -\chi'(\xi)$.

Substituting Eq. (2) into Eq. (1) and separating the real and imaginary parts lead to

$$k\rho + u\chi'\rho - a_1\chi'^2\rho + a_1\rho'' + a_2\rho^3 + a_3\rho^5 + a_4\rho^7 - a_5\chi'\rho^3 - a_7\chi'\rho^5 = 0, \tag{3}$$

and

$$-u\rho' + a_1\rho\chi'' + 2a_1\rho'\chi' + (3a_5 + 2a_6)\rho^2\rho' + (5a_7 + 4a_8)\rho^4\rho' = 0. \tag{4}$$

To solve these coupled equations, we introduce the ansatz

$$\chi' = \alpha\rho^4 + \beta\rho^2 + \gamma. \tag{5}$$

where α and β are the nonlinear chirp parameters, and γ is the constant chirp. Accordingly, the resultant chirp consisting of linear and nonlinear contributions can be obtained as

$$\delta\omega(t, z) = -(\alpha\rho^4 + \beta\rho^2 + \gamma) \tag{6}$$

The substitution of ansatz (5) into Eq. (4) yields three algebraic equations that define the chirp parameters:

$$\alpha = -\frac{5a_7 + 4a_8}{6a_1}, \quad \beta = -\frac{3a_5 + 2a_6}{4a_1}, \quad \gamma = \frac{u}{2a_1}. \tag{7}$$

Eq. (6) together with (7) is an important result; it shows that the chirping is expressed in powers of intensity I of the light pulse (where $I = |E|^2 = \rho^2$) as: $\delta\omega(t, z) = -(\alpha I^2 + \beta I + \gamma)$, and therefore its expression is different from that of the cubic–quintic NLS model considered in Ref. [31], in which only the two last terms are involved. This novel form of the phase possesses two intensity dependent chirping terms in addition to the simplest linear contribution. Note that these two terms appear due to higher-order nonlinearities and crucially affect the pulse dynamics. It follows from Eq. (6) that when higher-order nonlinearities manifest themselves in a material with higher-order nonlinear susceptibilities, the chirping associated with optical pulses is generally a nonlinear function of the intensity of the wave. We will show that the intensity dependence of $\delta\omega$ plays an important role in the investigation of chirped femtosecond pulse propagation in higher-order nonlinear media.

Now, using Eqs. (5) and (7) in Eq. (3), we obtain

$$\rho'' + b_1\rho^9 + b_2\rho^7 + b_3\rho^5 + b_4\rho^3 + b_5\rho = 0, \tag{8}$$

where

$$b_1 = \frac{(5a_7 + 4a_8)(a_7 - 4a_8)}{36a_1^2},$$
$$b_2 = \frac{3a_1a_4 + a_5(a_7 - a_8) - a_6(a_7 + 2a_8)}{3a_1^2},$$
$$b_3 = \frac{(16a_1a_3 - 8ua_7) - (2a_6 + 3a_5)(2a_6 - a_5)}{16a_1^2},$$
$$b_4 = \frac{2a_1a_2 - ua_5}{2a_1^2}, \quad b_5 = \frac{4ka_1 + u^2}{4a_1^2}. \tag{9}$$

Eq. (8) is an elliptic-type differential equation. Generally speaking, it is difficult to give the general solution of Eq. (8) when all the coefficients have nonzero values. Notice that in the particular case where $b_1 = b_2 = 0$, one finds chirped dark (bright) solitons and double-kink solitons of the cubic–quintic nonlinear Schrödinger equation with self-steepening and self-frequency shift, as presented in Ref. [31]. This means that the types of chirped solitonlike solutions presented here have a more general form than those in the earlier report [31]. Importantly, the presence of the parameters b_1 and b_2 permits the elliptic equation (8) to describe the features of chirped femtosecond solitons in the cubic–quintic–septic nonlinear Schrödinger equation with self-steepening, self-frequency shift, and non-Kerr quintic nonlinearities.

3. Chirped soliton-like solutions

In what follows, we present various chirped solitonlike solutions of the septic model (1), for different parameter conditions. Before discussing exact solutions to the elliptic equation (8), let us rewrite it in a more simplified form. Multiplying (8) by ρ' and integrating with respect to ξ, we get

$$(\rho')^2 + \frac{b_1}{5}\rho^{10} + \frac{b_2}{4}\rho^8 + \frac{b_3}{3}\rho^6 + \frac{b_4}{2}\rho^4 + b_5\rho^2 + b_6 = 0, \tag{10}$$

where b_6 is an integration constant. The change of variable for the field amplitude

$$\rho(\xi) = \sqrt{F(\xi)} \tag{11}$$

transforms Eq. (10) into the following new auxiliary elliptic equation:

$$\left(\frac{dF}{d\xi}\right)^2 = c_6F^6 + c_5F^5 + c_4F^4 + c_3F^3 + c_2F^2 + c_1F, \tag{12}$$

where $c_1 = -4b_6$, $c_2 = -4b_5$, $c_3 = -2b_4$, $c_4 = -\frac{4}{3}b_3$, $c_5 = -b_2$, and $c_6 = -\frac{4b_1}{5}$. This equation is much simpler than Eq. (10), so we now turn attention to solving the problem within the framework of this equation. The latter describes different types of envelope solutions, including soliton solutions, kink and antikink solutions, periodic and exponential solutions. Here, we are interested in the solitonlike solutions of this elliptic equation.

For Eq. (12), some restrictive special solutions of the bright, dark and solitonlike types have been obtained before. Notably, Yomba [39] presented some soliton solutions to the elliptic equation (12) in the particular case where $c_1 = c_3 = c_5 = 0$. Additionally, a number of solitonlike solutions have been listed in [39,40] when $c_1 = c_5 = c_6 = 0$ in (12). But there exist no known exact analytical soliton solutions to Eq. (12) with $c_i \neq 0$ in the literature. Here, for the first time to our knowledge, we present exact soliton solutions to the elliptic Eq. (12) in the general case when all the coefficients have nonzero values. This allows us to find chirped soliton solutions propagating in non-Kerr media with competing cubic–quintic–septic nonlinearity.

We first discuss some interesting specific cases for which some of the c_i coefficients in (12) are equal to 0. We start with the case where $c_1 = 0$ [i.e., the integration constant b_6 equal to zero], $c_3 = 0$ [i.e., the cubic term is related to self-steepening term], and $c_5 = 0$ [i.e., the septic term is related to self-steepening, self-frequency shift, and quintic non-Kerr terms]. Importantly, the case $c_1 = c_3 = c_5 = 0$ (or equivalently $b_2 = b_4 = b_6 = 0$) leads to the following three families of soliton solutions of the model equation (1):

(i) For $c_2 > 0$ [i.e., $b_5 < 0$], which implies $k < -\frac{u^2}{4a_1}$, one obtains a bright soliton solution of (12) of the form [39]

$$F(\xi) = \left(\frac{-c_2c_4\,\mathrm{sech}^2(\sqrt{c_2}\,\xi)}{c_4^2 - c_2c_6\left(1 + \epsilon\tanh(\sqrt{c_2}\,\xi)\right)^2}\right)^{1/2}, \tag{13}$$

where $\epsilon = \pm 1$. Substituting (13) along with relation (11) into (2), we obtain a bright soliton solution of the form:

$$E(z, t) = \left(\frac{-c_2c_4\,\mathrm{sech}^2(\sqrt{c_2}(t - uz))}{c_4^2 - c_2c_6\left(1 + \epsilon\tanh(\sqrt{c_2}(t - uz))\right)^2}\right)^{1/4} e^{i[\chi(\xi) - kz]}. \tag{14}$$

The corresponding chirping is given by

$$\delta\omega(t, z) = \frac{\alpha c_2c_4\,\mathrm{sech}^2(\sqrt{c_2}\,\xi)}{c_4^2 - c_2c_6\left(1 + \epsilon\tanh(\sqrt{c_2}\,\xi)\right)^2} - \beta\left(\frac{-c_2c_4\,\mathrm{sech}^2(\sqrt{c_2}\,\xi)}{c_4^2 - c_2c_6\left(1 + \epsilon\tanh(\sqrt{c_2}\,\xi)\right)^2}\right)^{1/2} - \gamma. \tag{15}$$

As seen, the first two terms in Eq. (15) denote the nonlinear chirp that results from the quintic non-Kerr nonlinearity, self-steepening, and self-frequency-shift, while the last term accounts for the linear chirp.

Since $c_2 > 0$, we require $c_4(c_4^2 - c_2c_6) < 0$ for the soliton solution (14) and the chirp (15) to exist. This leads to the constraint condition $b_3(5b_3^2 - 9b_1b_5) > 0$. We also have the restriction that $b_2 = b_4 = b_6 = 0$. These constraint conditions represent the strict balances among non-Kerr nonlinearity, cubic–quintic–septic nonlinearities, self-steepening, and delayed nonlinear response effects.

Fig. 1 depicts intensity profile of a typical bright soliton for the following values of the model parameters: $a_1 = 1.6001$, $a_2 = -2.6885$, $a_4 = 0.1164$, $a_5 = 0.30814$, $a_6 = 0.76604$, $a_7 = 0.011$, $k = -150.2856$. To make $b_2 = b_4 = 0$, we set $u = -27.9215$ and $a_8 = 0.30089$. Furthermore, to make $b_3(5b_3^2 - 9b_1b_5) > 0$ we set $a_3 = 1.218$. The corresponding chirping for the bright soliton is shown in Fig. 2 for z=0.

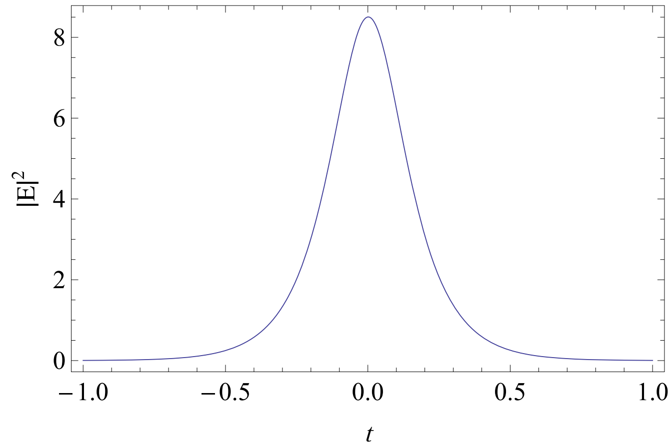

Fig. 1. Intensity profile of the soliton solution given by Eq. (14) at z=0 for the values mentioned in the text.

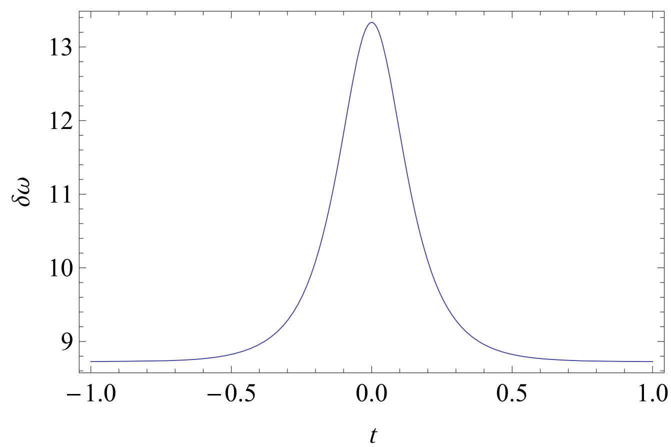

Fig. 2. Chirping profile for the bright soliton plotted in Fig. 1.

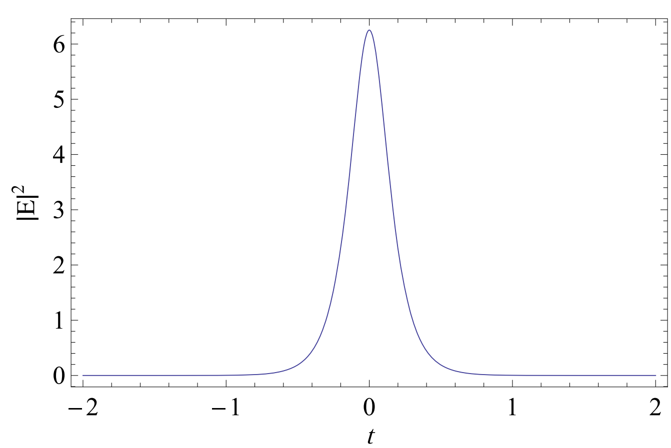

Fig. 3. Intensity profile of the soliton solution given by Eq. (17) at z=0 for the values mentioned in the text.

(ii) For $c_2 > 0$ and $\Delta = c_4^2 - 4c_2c_6 > 0$ [i.e., $b_5 < 0$ and $b_3^2 > \frac{36b_1b_5}{5}$], one finds another bright-soliton-type solution for (12) of the form [39]

$$F(\xi) = \left(\frac{2c_2}{\epsilon\sqrt{\Delta}\,\cosh\left(2\sqrt{c_2}\,\xi\right) - c_4}\right)^{1/2}, \tag{16}$$

where $\epsilon = \pm 1$. We notice that the imposed conditions imply that one must have $k < -\frac{u^2}{4a_1}$ and

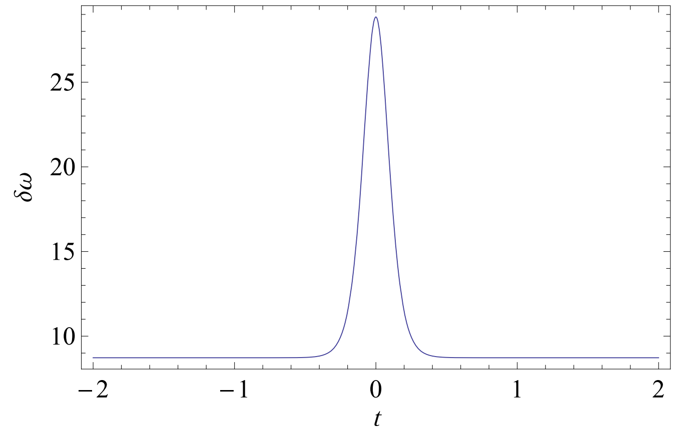

Fig. 4. Chirping profile for the bright soliton plotted in Fig. 3.

$[(16a_1a_3 - 8ua_7) - (2a_6 + 3a_5)(2a_6 - a_5)]^2 > \frac{64}{5}(5a_7 + 4a_8)(a_7 - 4a_8)(4ka_1 + u^2)$. Then, substituting solution (16) along with relation (11) into (2), we obtain another bright-type soliton solution of (1) in the form

$$E(z,t) = \left(\frac{2c_2}{\epsilon\sqrt{\Delta}\,\cosh\left(2\sqrt{c_2}\,(t-uz)\right) - c_4}\right)^{1/4} e^{i[\chi(\xi)-kz]}, \tag{17}$$

for which the chirping is given by

$$\delta\omega(t,z) = \frac{-2\alpha c_2}{\epsilon\sqrt{\Delta}\,\cosh\left(2\sqrt{c_2}\,\xi\right) - c_4} - \beta\left(\frac{2c_2}{\epsilon\sqrt{\Delta}\,\cosh\left(2\sqrt{c_2}\,\xi\right) - c_4}\right)^{1/2} - \gamma. \tag{18}$$

The intensity profile of this kind of bright soliton is shown in Fig. 3. Here we have used the model parameters are $a_1 = 1.6001$, $a_2 = -2.6885$, $a_4 = 0.1164$, $a_5 = 0.30814$, $a_6 = 0.76604$, $a_7 = 0.011$, $k = -150.2856$, $u = -27.9215$ and $a_8 = 0.30089$. To make $b_3^2 > \frac{36b_1b_5}{5}$, we set $a_3 = 2.36$. We also display the corresponding pulse chirp in Fig. 4.

(iii) For $c_2 > 0$ and $\Delta = c_4^2 - 4c_2c_6 = 0$ [i.e., $b_5 < 0$ and $b_3^2 = \frac{36b_1b_5}{5}$], Eq. (12) admits a kink-type solution of the form [39]

$$F(\xi) = \left(-\frac{c_2}{c_4}\left(1 + \epsilon\tanh\left(\frac{\sqrt{c_2}}{2}\xi\right)\right)\right)^{1/2}, \tag{19}$$

where $\epsilon = \pm 1$. Then, substituting solution (19) along with relation (11) into (2), we have a chirped kink type soliton solution of (1) as follows

$$E(z,t) = \left(-\frac{c_2}{c_4}\left(1 + \epsilon\tanh\left(\frac{\sqrt{c_2}}{2}(t-uz)\right)\right)\right)^{1/4} e^{i[\chi(\xi)-kz]}. \tag{20}$$

If we insert solution (19) along with relation (11) into (6), we find that the chirping is given by

$$\delta\omega(t,z) = \frac{\alpha c_2}{c_4}\left(1 + \epsilon\tanh\left(\frac{\sqrt{c_2}}{2}(t-uz)\right)\right) - \beta\left(-\frac{c_2}{c_4}\left(1 + \epsilon\tanh\left(\frac{\sqrt{c_2}}{2}(t-uz)\right)\right)\right)^{1/2} - \gamma \tag{21}$$

Here, the requirement that $b_5 < 0$ and $b_3^2 = \frac{36b_1b_5}{5}$ translate into restrictions on the model coefficients, namely $k < -\frac{u^2}{4a_1}$ and

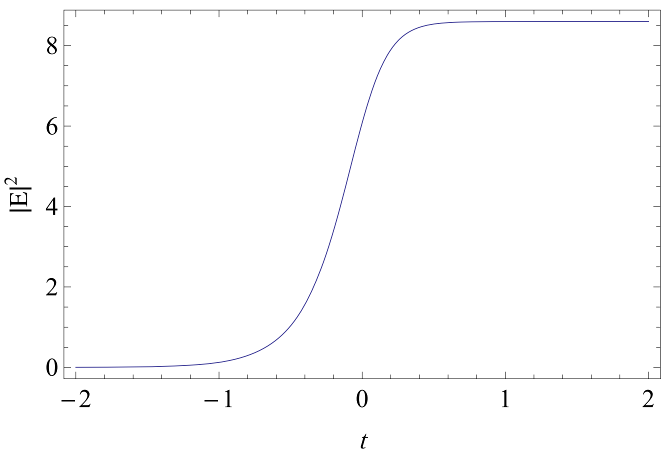

Fig. 5. Intensity profile of the soliton solution given by Eq. (20) at $z=0$ for the values mentioned in the text.

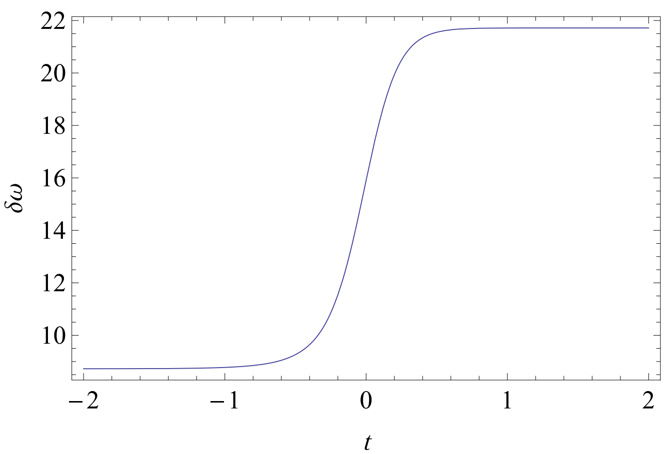

Fig. 6. Chirping profile for the kink soliton plotted in Fig. 5.

$[(16a_1a_3 - 8ua_7) - (2a_6 + 3a_5)(2a_6 - a_5)]^2$. Also, one must require $= \frac{64}{5}(5a_7 + 4a_8)(a_7 - 4a_8)(4ka_1 + u^2)$
$c_4 < 0$ [i.e., $b_3 > 0$] for the solution (20) and its corresponding nonlinear chirp (21) to exist, since $c_2 > 0$.

Fig. 5 shows intensity profile of a typical kink solution. The values of the parameters are $a_1 = 1.6001$, $a_2 = -2.6885$, $a_4 = 0.1164$, $a_5 = 0.30814$, $a_6 = 0.76604$, $a_7 = 0.011$, $k = -150.2856$, $u = -27.9215$ and $a_8 = 0.30089$. To make $b_3^2 = \frac{36b_1b_5}{5}$, we set $a_3 = 2.3328$. Fig. 6 illustrates the chirp associated with this soliton solution.

Next, we study exact solitonlike solutions to the elliptic equation (12) in the limiting case $c_1 = c_5 = c_6 = 0$. This leads to the condition $b_1 = b_2 = b_6 = 0$. For this case, Eq. (12) possesses an exact soliton solution of the form

$$F(\xi) = \frac{2c_2\,\mathrm{sech}(\sqrt{c_2}\,\xi)}{\sqrt{c_3^2 - 4c_2c_4} - c_3\,\mathrm{sech}\left(\sqrt{c_2}\,\xi\right)}, \tag{22}$$

with $c_2 > 0$ and $c_3^2 - 4c_2c_4 > 0$. Note that this solution was found earlier by Sirendaoreji and Jiong [40]. By inserting this expression into (2) and using (11), we obtain a bright-type soliton solution of (1) in the form

$$E(z,t) = \left(\frac{2c_2\,\mathrm{sech}\left(\sqrt{c_2}\,(t-uz)\right)}{\sqrt{c_3^2 - 4c_2c_4} - c_3\,\mathrm{sech}\left(\sqrt{c_2}\,(t-uz)\right)}\right)^{1/2} e^{i[\chi(\xi) - kz]}. \tag{23}$$

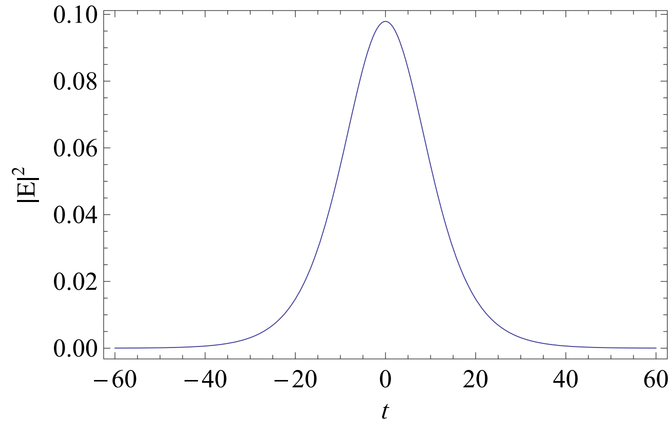

Fig. 7. Intensity profile of the soliton solution given by Eq. (23) at $z=0$ for the values mentioned in the text.

This solution is valid only if $c_2 > 0$ and $c_3^2 - 4c_2c_4 > 0$, or equivalently we have the restrictions that $b_5 < 0$ and $b_4^2 > \frac{16}{3}b_3b_5$. In terms of the model coefficients, we find that this solution exists when $k < -\frac{u^2}{4a_1}$ and $(2a_1a_2 - ua_5)^2 > \frac{1}{3}[(16a_1a_3 - 8ua_7) - (2a_6 + 3a_5)(2a_6 - a_5)]$.
$(4ka_1 + u^2)$

The associated chirp takes the expression:

$$\delta\omega(t,z) = -\alpha\left(\frac{2c_2\,\mathrm{sech}(\sqrt{c_2}\,\xi)}{\sqrt{c_3^2 - 4c_2c_4} - c_3\,\mathrm{sech}\left(\sqrt{c_2}\,\xi\right)}\right)^2 - \frac{2\beta c_2\,\mathrm{sech}(\sqrt{c_2}\,\xi)}{\sqrt{c_3^2 - 4c_2c_4} - c_3\,\mathrm{sech}\left(\sqrt{c_2}\,\xi\right)} - \gamma \tag{24}$$

Fig. 7 presents the intensity profile of a typical bright soliton for parameter values: $a_1 = 1.6001$, $a_2 = -2.6885$, $a_3 = 0.32$, $a_5 = 0.30814$, $a_6 = 0.76604$, $k = -141.8290$, $u = -30.1280$, and $a_7 = -0.0247$. To make $b_1 = 0$, we set $a_8 = -0.006175$. Furthermore, to make $b_2 = 0$ we set $a_4 = -0.00472$. The corresponding pulse chirp obtained for the same values of parameters is shown in Fig. 8.

It is physically interesting to investigate other forms of kink solution of the septic model (1), for special choices of parameters. Here we discuss this type of pulse envelope by setting $c_1 = c_3 = c_5 = 0$ in Eq. (12). The main reason for pursuing such a goal would be the possible applications of this type of solitonic solutions to long-distance optical communication systems, taking advantage of their particular properties. In this case, if we use the change of variable $F^2 = u$, the wave function $u(\xi)$ satisfies the

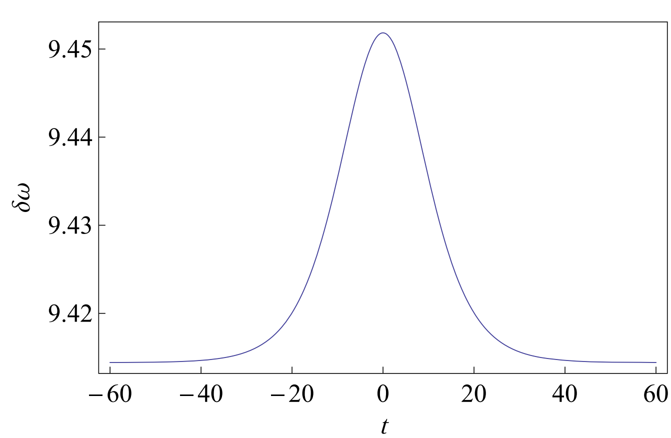

Fig. 8. Chirping profile for the bright soliton plotted in Fig. 7.

following nonlinear ordinary differential equation:

$$(u')^2 = 4c_6u^4 + 4c_4u^3 + 4c_2u^2. \tag{25}$$

Using a fractional transformation [41]

$$u(\xi) = \frac{A + Bf(\xi, m)}{1 + Df(\xi, m)} \tag{26}$$

brings solutions in the rational form, where A, B and D are real constants, and $f(\xi, m)$ is a Jacobi elliptic function, with the modulus parameter m.

Here we focus on wave solutions of the form $f(\xi, m) = \mathrm{sn}(\xi, m)$, where $\mathrm{sn}(\xi, m)$ is the Jacobi elliptic sine function. If we insert solution (26) into (25), we obtain

$$\begin{aligned}
&(B - AD)^2[1 - \mathrm{sn}^2(\xi) - m^2\,\mathrm{sn}^2(\xi) + m^2\,\mathrm{sn}^4(\xi)] \\
&\quad - 4c_6\{A^4 + B^4\,\mathrm{sn}^4(\xi) + 4AB^3\,\mathrm{sn}^3(\xi) + 4A^3B\,\mathrm{sn}(\xi) \\
&\quad + 6A^2B^2\,\mathrm{sn}^2(\xi)\} \\
&\quad - 4c_4\{A^3 + B^3\,\mathrm{sn}^3(\xi) + 3AB^2\,\mathrm{sn}^2(\xi) + 3A^2B\,\mathrm{sn}(\xi) \\
&\quad + A^3D\,\mathrm{sn}(\xi) + B^3D\,\mathrm{sn}^4(\xi) + 3AB^2D\,\mathrm{sn}^3(\xi) \\
&\quad + 3A^2BD\,\mathrm{sn}^2(\xi)\} \\
&\quad - 4c_2\{A^2 + B^2\,\mathrm{sn}^2(\xi) + 2AB\,\mathrm{sn}(\xi) + A^2D^2\,\mathrm{sn}^2(\xi) \\
&\quad + B^2D^2\,\mathrm{sn}^4(\xi) + 2ABD^2\,\mathrm{sn}^3(\xi) + 2A^2D\,\mathrm{sn}(\xi) \\
&\quad + 2B^2D\,\mathrm{sn}^3(\xi) + 4ABD\,\mathrm{sn}^2(\xi)\} = 0.
\end{aligned} \tag{27}$$

Matching the coefficients of different powers of sn function, one obtains

$$(B - AD)^2 - 4c_6A^4 - 4c_4A^3 - 4c_2A^2 = 0, \tag{28}$$

$$4c_6A^3B + c_4(3A^2B + A^3D) + 2c_2(AB + A^2D) = 0, \tag{29}$$

$$\begin{aligned}
&(1 + m^2)(B - AD)^2 + 24c_6A^2B^2 + 12c_4(AB^2 + A^2BD) \\
&\quad + 4c_2(B^2 + A^2D^2 + 4ABD) = 0,
\end{aligned} \tag{30}$$

$$4c_6AB^3 + c_4(B^3 + 3AB^2D) + 2c_2(ABD^2 + B^2D) = 0, \tag{31}$$

$$(B - AD)^2m^2 - 4c_6B^4 - 4c_4B^3D - 4c_2B^2D^2 = 0. \tag{32}$$

Solving the above equations one gets

$$B = \lambda D, \tag{33}$$

where

$$\lambda = \frac{A(1 + m^2) - 8c_2A}{(1 + m^2) + 6c_4A + 4c_2},$$

$$D = \frac{2A}{\lambda - A}\left(c_6A^2 + c_4A + c_2\right)^{1/2}, \tag{34}$$

and

$$A = \frac{-\left[c_4(1 + m^2) - 2c_2c_4\right] \pm \sqrt{\left[c_4(1 + m^2) - 2c_2c_4\right]^2 - c_2\left\{4c_6\left[(1 + m^2) - 8c_2\right] + 6c_4^2\right\}\left[(1 + m^2) - 2c_2\right]}}{2c_6\left[(1 + m^2) - 8c_2\right] + 3c_4^2}, \tag{35}$$

under the condition: $16A^2\lambda^2[c_6A^2 + c_4A + c_2][c_6\lambda^2 + c_4\lambda + c_2] = (\lambda - A)^4m^2$. From these results, one can write the new exact periodic wave solution of Eq. (1) as

$$E(z, t) = \left(\frac{A + B\,\mathrm{sn}(\xi)}{1 + D\,\mathrm{sn}(\xi)}\right)^{1/4} e^{i[\chi(\xi) - kz]}, \tag{36}$$

When $m \to 1$, then $\mathrm{sn}(\xi) = \tanh(\xi)$ and the solution (36) becomes a kink-type solution of the form

$$E(z, t) = \left(\frac{A + B\tanh(\xi)}{1 + D\tanh(\xi)}\right)^{1/4} e^{i[\chi(\xi) - kz]}, \tag{37}$$

where the constants A, B, and D are given by relations (33)–(35), with modulus parameter $m=1$. It is important to notice that the kink solution (37) may approach nonzero value when the ξ variable approaches infinity. Such envelope solution cannot exist in the cubic NLS equation.

The accompanying chirping is given by

$$\delta\omega(t, z) = -\alpha\left(\frac{A + B\tanh(\xi)}{1 + D\tanh(\xi)}\right) - \beta\left(\frac{A + B\tanh(\xi)}{1 + D\tanh(\xi)}\right)^{1/2} - \gamma, \tag{38}$$

provided that $c_1 = c_3 = c_5 = 0$, or equivalently the constants $b_2 = b_4 = b_6 = 0$.

Let us now consider the most general case when all the c_i coefficients in (12) have nonzero values. This is an important case, in which the competing cubic–quintic–septic nonlinearities and non-Kerr terms act jointly with the self-steepening and self-frequency shift, taken at the lowest order. Here, in the general case, we have to solve Eq. (12) without setting the c_i coefficients to zero. To this end, we introduce the following ansatz for $F(\xi)$:

$$F(\xi) = a + b\,\mathrm{sech}^{1/2}(\eta\xi), \tag{39}$$

where a, b and η are unknown parameters to be determined. Substituting the ansatz, Eq. (39) into Eq. (12) we obtain

$$\begin{aligned}
&\tfrac{1}{4}b^2\eta^2\,\mathrm{sech}(\eta\xi) - \tfrac{1}{4}b^2\eta^2\,\mathrm{sech}^3(\eta\xi) \\
&\quad - c_6\{a^6 + b^6\,\mathrm{sech}^3(\eta\xi) + 6ab^5\,\mathrm{sech}^{5/2}(\eta\xi) \\
&\quad + 6a^5b\,\mathrm{sech}^{1/2}(\eta\xi) + 15a^2b^4\,\mathrm{sech}^2(\eta\xi) + 15a^4b^2\,\mathrm{sech}(\eta\xi) \\
&\quad + 20a^3b^3\,\mathrm{sech}^{3/2}(\eta\xi)\} \\
&\quad - c_5\{a^5 + b^5\,\mathrm{sech}^{5/2}(\eta\xi) + 5a^4b\,\mathrm{sech}^{1/2}(\eta\xi) \\
&\quad + 5ab^4\,\mathrm{sech}^2(\eta\xi) + 10a^2b^3\,\mathrm{sech}^{3/2}(\eta\xi) \\
&\quad + 10a^3b^2\,\mathrm{sech}(\eta\xi)\} \\
&\quad - c_4\{a^4 + b^4\,\mathrm{sech}^2(\eta\xi) + 4a^3b\,\mathrm{sech}^{1/2}(\eta\xi) \\
&\quad + 4ab^3\,\mathrm{sech}^{3/2}(\eta\xi) + 6a^2b^2\,\mathrm{sech}(\eta\xi)\} \\
&\quad - c_3\{a^3 + b^3\,\mathrm{sech}^{3/2}(\eta\xi) + 3a^2b\,\mathrm{sech}^{1/2}(\eta\xi) \\
&\quad + 3ab^2\,\mathrm{sech}(\eta\xi)\} \\
&\quad - c_2\{a^2 + b^2\,\mathrm{sech}(\eta\xi) + 2ab\,\mathrm{sech}^{1/2}(\eta\xi)\} \\
&\quad - c_1\{a + b\,\mathrm{sech}^{1/2}(\eta\xi)\} = 0.
\end{aligned} \tag{40}$$

Equating the powers of sech we obtain the conditions

$$c_6a^6 + c_5a^5 + c_4a^4 + c_3a^3 + c_2a^2 + c_1a = 0, \tag{41}$$

$$\frac{1}{4}b^2\eta^2 - 15c_6a^4b^2 - 10c_5a^3b^2 - 6c_4a^2b^2 - 3c_3ab^2 - c_2b^2 = 0, \tag{42}$$

$$-15c_6a^2b^4 - 5c_5ab^4 - c_4b^4 = 0, \tag{43}$$

$$-\frac{1}{4}b^2\eta^2 - c_6b^6 = 0, \tag{44}$$

$$-6c_6ab^5 - c_5b^5 = 0, \tag{45}$$

$$-6c_6a^5b - 5c_5a^4b - 4c_4a^3b - 3c_3a^2b - 2c_2ab - c_1b = 0,$$

$$-20c_6a^3b^3 - 10c_5a^2b^3 - 4c_4ab^3 - c_3b^3 = 0, \tag{46}$$

Solving the algebraic equations obtained above, one finds

$$a = -\frac{c_5}{6c_6}, \tag{47}$$

$$b = \left(-\frac{30c_4c_6c_5^2 + 216c_2c_6^3 - 5c_5^4 - 108c_3c_5c_6^2}{216c_6^4}\right)^{1/4}, \tag{48}$$

$$\eta = \left(\frac{30c_4c_6c_5^2 + 216c_2c_6^3 - 5c_5^4 - 108c_3c_5c_6^2}{54c_6^3}\right)^{1/2}, \tag{49}$$

with the condition on parameters

$$8c_4c_5^2c_6 - c_5^4 + 24c_6^2(6c_6c_2 - 2c_3c_5) + 6c_6^2c_5c_3 = 0. \tag{50}$$

By substituting solution (39) along with relation (11) into (2), we obtain an exact chirped soliton solution of (1) in the form

$$E(z, t) = \left(a + b\,\mathrm{sech}^{1/2}\left(\eta(t - uz)\right)\right)^{1/2} e^{i[\chi(\xi) - kz]}, \tag{51}$$

where the parameters a, b and η are determined using Eqs. (47)–(49).

It is easy to see that the soliton solution (51) can have either bright or dark intensity profiles that uniquely depend on the signs of the parameters a and b. When $a > 0$ and $b > 0$, this solution will be of the bright type. However, if one or both of these parameters have negative values, the solution will be of the dark-type. The main property of this soliton solution is that it has a characteristic platform underneath it. The chirping corresponding to this solution is given by

$$\delta\omega(t, z) = -\alpha\left(a + b\,\mathrm{sech}^{1/2}\left(\eta(t - uz)\right)\right)^2 - \beta\left(a + b\,\mathrm{sech}^{1/2}\left(\eta(t - uz)\right)\right) - \gamma. \tag{52}$$

Because parameter η needs to be positive for the existence of soliton solution (51) and its corresponding chirping (52), one must choose the parameters c_i to satisfy $c_6^3(30c_4c_6c_5^2 + 216c_2c_6^3 - 5c_5^4 - 108c_3c_5c_6^2) > 0$, as it follows from (49). At the same time, for the existence of this exact soliton solution, the constraint (50) must be satisfied. Hence, for the general case, the relationship among the parameters is much more complicated.

Finally, we mention that other types of complex localized structures, such as the nonfundamental soliton structures (e.g., dipoles and multipoles), can be supported by the higher-order nonlinearities. We have also found that these pulses are possible in a more complex model of the HNLS equation, which involves both of high-order dispersion (e.g., third- and fourth-order dispersions) and nonlinear effects [33,42]. We note in passing that for a high-order nonlinear class of systems, an experiment has recently been reported in glassy materials, such as chalcogenide glass, which exhibits both of third-, fifth-, and seventh-order nonlinearities [43].

4. Conclusion

In this paper, we have obtained exact chirped soliton solutions of a higher-order nonlinear Schrödinger equation, modeling the propagation of an ultrashort femtosecond optical pulses. The model used combines cubic, quintic and septic nonlinearities, non-Kerr nonlinear terms, as well as self-steepening and self-frequency shift. After introducing a new ansatz that includes a novel form of chirping, the solutions were investigated within the framework of a general elliptic equation involving many parameters. This allows finding a rich set of chirped soliton solutions for the governing equation. These optical pulses exhibit explicitly a nonlinear chirp that arises from higher-order nonlinear effects, and differs from that in Ref. [31] in having an additional highly intensity-dependent term. The existence domain for the chirped soliton solutions has been found in the HNLS equation parameter set. Furthermore, a new ansatz is introduced for the construction of exact chirped soliton solution for the higher order wave equation without any constraints on the parameters of the resultant elliptic equation. The obtained results show that the interplay between higher-order nonlinearities and non-Kerr terms leads to new interesting chirped soliton solutions. These chirped femtosecond solitons may find straightforward applications in higher-order nonlinear fiber systems, wherein the effect of cubic–quintic–septic nonlinearities as well as of quintic non-Kerr terms should be included.

We intend to extend our study of the cubic–quintic–septic nonlinear Schrödinger equation with non-Kerr terms to the same model containing distributed coefficients, to see how the variation of dispersion and nonlinearity along the propagation direction of the fiber affects the nonlinear chirp associated with solitonic solutions. Such a study would be very important, as the generalized nonlinear Schrödinger equation with varying dispersion and nonlinearity is required for describing dispersion-management or soliton control. Possible extension to coupled nonlinear Schrödinger equations would also be interesting, because such models have found applications in various settings, especially in nonlinear optics and in the dynamics of Bose–Einstein condensates.

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