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Two-dimensional accessible solitons in *PT*-symmetric potentials

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Abstract Two-dimensional parity-time (*PT*) symmetric potentials are introduced, which allow the existence of spatial solitons in the model of the strongly nonlocal nonlinear Schrödinger equation. Two-dimensional accessible solitons are found in the form of solutions separating the radial amplitude, given in terms of Laguerre polynomials, a phase function involving quadratic, linear, and constant phase shifts, and a specific azimuthal modulation function. Shape-preserving solitons are constructed from Laguerre–Gaussian functions containing the self-similar variable and an exponential form of the azimuthal modulation, containing sine and cosine functions, when a suitable *PT*-symmetric potential is chosen. Interesting soliton profiles and the corresponding *PT*-symmetric potentials are displayed for different values of the parameters.

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1 Introduction

In the past decade, beam propagation in optical media with parity-time (*PT*) symmetry has been widely studied, both theoretically and experimentally [1–3]. The motivation came from the initial ideas of generalization of quantum mechanics, suggested by Bender et al. [4, 5], with *PT*-symmetric complex potentials [2]. Coupling the parity symmetry ($\hat{P}$) with the time reversal symmetry ($\hat{T}$) has produced a wide class of non-Hermitian Hamiltonian systems that have drawn significant attention: similar to traditional quantum mechanics; these non-Hermitian Hamiltonians still possess real eigenvalue spectra [5] and may have physical meaning. This has stimulated a series of applications of *PT* symmetry in various areas of physics, ranging from *PT*-symmetric harmonic and anharmonic oscillators, linear and nonlinear optics, to quantum field theory [5–15]. An increased interest in *PT*-symmetric systems has been generated, because of the unexpected physical properties, such as double refraction, phase transition, and the appearance of curious optical solitons [1, 6, 9, 10]. On the topic of solitons in *PT*-symmetric systems, Musslimani et al. [9] have studied localized beam solutions in *PT*-periodic potentials. The necessary condition of a Hamiltonian to possess *PT* symmetry has also been pointed out in their research work: the real part of a *PT* complex potential has to be an even function of position, whereas the imaginary part of that potential has to be an odd function of position. Applications of *PT*-symmetry to linear and nonlinear lattices have been undertaken in [11–13], to double-channel waveguides in [14], and to nonlocal nonlinearity in [15].

Optical spatial solitons [16] in nonlocal nonlinear (NNL) media have also received much attention lately [17–23], due to their rich potential for applications, such as photonic switching, all-optical switching and logic gating, and all optical signal processing. Optical spatial solitons are self-trapped optical beams that exist by virtue of the balance between diffraction and nonlinearity. The propagation of optical beams in NNL media is modeled by the NNL Schrödinger (NNLS) equation [17, 20]. Snyder and Mitchell [17] have simplified the NNLS equation in the case of strongly nonlocal media to a linear model, named the Snyder–Mitchell model, and they found an exact Gaussian-shaped solution, called the accessible soliton [17, 18]. Subsequently, Assanto et al. have observed accessible solitons in nematic liquid crystals (NLCs) [19, 20], called them nematicons, and proved theoretically [19] and experimentally [20] that NLCs indeed are some of the strongly NNL materials. Various phenomena connected with the propagation of three-dimensional (3D) solitons in NNL media have been addressed in [21–23]. We have also studied two- and three-dimensional NNLS equation in strongly NNL media [24–27], and obtained exact solutions in terms of 2D Laguerre–Gaussian soliton family [24], as well as Whittaker and Hermite–Gaussian functions [25]; we also found 2D and 3D optical vortex and necklace solitons in highly NNL media [26, 27].

In the present paper, we study 2D NNLS equation in the strongly NNL media with suitably chosen *PT*-symmetric potentials. The Laguerre–Gaussian solitons are constructed analytically, and we find that these solitons display many new interesting features. We utilize a two-dimensional *PT*-symmetric potential whose real part is an even function of the azimuthal angle and the imaginary part is an odd function of the azimuthal angle. Two-dimensional accessible solitons are found in the form of solutions separating the variables. These solitons are constructed using Laguerre polynomials, given in terms of the self-similar variable, and an exponential form of the azimuthal function, involving sine and cosine functions. Since our aim is to display 2D accessible solitons, we choose by construction the amplitude of the imaginary part of our *PT*-symmetric potential an order of magnitude smaller than the amplitude of the real part. In this manner, we stay away from and do not observe any indication of a phase transition in our model; our solutions remain stable and within the same class. The questions pertaining to the appearance of phase transition and (in)stability of solutions for larger values of the imaginary part will be addressed elsewhere.

The paper is organized as follows. In Sect. 2, the 2D Snyder–Mitchell model with a *PT*-symmetric potential is introduced, and an exact solution is constructed. In Sect. 3, we illustrate and discuss some examples of the exact solutions obtained. Section 4 gives the conclusion.

2 Model and analytical soliton solutions

In strongly NNL media with a *PT*-symmetric potential $V(r,\varphi)$, the 2D optical beam evolution is governed by the following scaled NNLS equation for the dimensionless light field $u(z,r,\varphi)$ [24, 25]:

$$i\frac{\partial u}{\partial z}+\frac{1}{2}\nabla_{\perp}^{2}u-sr^{2}-V(r,\varphi)u=0, \tag{1}$$

where z is the dimensionless propagation distance, $\nabla_{\perp}^{2}$ is the transverse Laplacian, and s is the parameter depending on the beam power P. Note that P is a constant of motion, equal to the total input power P_0. The second term in Eq. (1) represents the diffraction, the third term originates from the optical nonlocality, and the fourth term is the external potential function. To analyze the effects of *PT* symmetry on Eq. (1), we introduce the *PT*-symmetric potential function $V(r,\varphi)$. When $V(r,\varphi)=0$, Eq. (1) is simplified to the general 2D NNLS equation in strongly NNL media [24] representing a 2D quantum-mechanical harmonic oscillator. Since the parabolic potential is an even function of r, it is the external potential $V(r,\varphi)$ that decides whether Eq. (1) is *PT*-symmetric or not. It should also be noted that the beam collapse cannot occur in Eq. (1) with real V, because it is a linear equation. However, complex V introduces regions of gain and loss, in which the solution may grow or attenuate. The condition for *PT*-symmetry in Eq. (1) is that [28]

$$V^{*}(r,\varphi+\pi)=V(r,\varphi). \tag{2}$$

We treat Eq. (1) in polar coordinates, by the self-similar method and the separation of variables, and consider only the case $s > 0$. In polar coordinates, the transverse Laplacian is $\nabla_\perp^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$, where $r = \sqrt{x^2 + y^2}$ is the distance from the z axis, and φ is the azimuthal angle. Following [24, 26], we define the complex field as

$$u(z, r, \varphi) = A(z, r)\Phi(\varphi)e^{iB(z,r)}, \tag{3}$$

where the amplitude $A(z, r)$ and the phase $B(z, r)$ are real functions of z and r. The azimuthal part of the solution $\Phi(\varphi)$ satisfies the following equation:

$$\frac{1}{\Phi}\frac{\partial^2\Phi}{\partial\varphi^2} = \Omega(\varphi), \tag{4}$$

where $\Omega(\varphi) \neq 0$; otherwise Eq. (1) is not a *PT*-symmetric system. Evidently, we may choose *PT*-symmetric potentials in the form:

$$V(r, \varphi) = \frac{\Omega(\varphi)}{2r^2}, \tag{5}$$

where $\Omega(\varphi)$ should satisfy Eq. (2), namely

$$\Omega^*(\varphi + \pi) = \Omega(\varphi). \tag{6}$$

Thus, considering the function $\Omega(\varphi)$ is equivalent to considering the *PT*-symmetric potential $V(r, \varphi)$. Therefore, we study in some detail the function $\Omega(\varphi)$ in the following text. It is also important to note that Eq. (6) imposes different constraints on the real and imaginary components of $\Omega(\varphi)$: $\Omega_R(\varphi + \pi) = \Omega_R(\varphi)$ and $\Omega_I(\varphi + \pi) = -\Omega_I(\varphi)$. Hence, for the periodic $\Omega(\varphi)$ this means that the periodicity of the two components has to be different. All along, Eq. (4) must be satisfied for the azimuthal part $\Phi(\varphi)$.

In this paper, we consider the following special solution:

$$\Phi(\varphi) = ke^{iI_0 \sin[(2m+1)\varphi] + \cos(2m\varphi)}, \tag{7}$$

where the normalization constant

$$k = \frac{1}{\sqrt{\int_0^{2\pi} e^{2\cos(2m\varphi)}\, d\varphi}},$$

I_0 is the strength of the imaginary part (the strength of the real part is chosen to be 1), and m $(= 0, 1, 2, \ldots)$ is a nonnegative integer. Different azimuthal solutions, coming from different choices of the $\Omega(\varphi)$ functions, lead to different classes of solitons. Substituting Eq. (7) into Eq. (4), we find the corresponding $\Omega(\varphi)$ function:

$$\Omega(\varphi) = \Omega_R(\varphi) + i\Omega_I(\varphi), \tag{8}$$

with

$$\begin{aligned}
\Omega_R(\varphi) = {} & -4m^2 \cos(2m\varphi) - \frac{1}{2}(2m + 1)^2 I_0^2 \\
& - \frac{1}{2}(2m + 1)^2 I_0^2 \cos\bigl[2(2m + 1)\varphi\bigr] \\
& + 4m^2 \sin^2(2m\varphi), \\
\Omega_I(\varphi) = {} & -4(2m + 1)mI_0 \cos\bigl[(2m + 1)\varphi\bigr] \sin(2m\varphi) \\
& - (2m + 1)^2 I_0 \sin\bigl[(2m + 1)\varphi\bigr].
\end{aligned}$$

Evidently, Eq. (8) satisfies the *PT*-symmetry condition (6). Thus, we pick a form of the azimuthal part of the solution, consistent with the form for the *PT*-symmetric potential function $\Omega(\varphi)$ that supports such a solution through Eq. (4).

Substituting Eq. (3) into Eq. (1) and considering Eq. (5), one requires that the real and imaginary parts of each term be separately equal to zero. In this manner, one obtains a set of coupled equations for A and B:

$$\frac{1}{A}\frac{\partial A}{\partial z} + \frac{1}{A}\frac{\partial A}{\partial r}\frac{\partial B}{\partial r} + \frac{1}{2}\frac{\partial^2 B}{\partial r^2} + \frac{1}{2r}\frac{\partial B}{\partial r} = 0, \tag{9a}$$

$$-\frac{\partial B}{\partial z} + \frac{1}{2A}\frac{\partial^2 A}{\partial r^2} - \frac{1}{2}\left(\frac{\partial B}{\partial r}\right)^2 + \frac{1}{2rA}\frac{\partial A}{\partial r} - sr^2 = 0. \tag{9b}$$

We search for the self-similar solution of Eqs. (9a) and (9b); to this end, we presume the following form of the solutions [24, 25]: $A = \frac{1}{w(z)}F(\theta)$ and $B(z, r) = a(z)r^2 + b(z)r + c(z)$, with clear meaning of the parameter functions. Here, $w(z)$ is the beam width, $\theta(z, r)$ is the self-similar variable, and $a(z)$, $b(z)$, $c(z)$ are the quadratic, linear, and constant parameter functions of the phase. They are all allowed to vary with the propagation distance z. Inserting these transformations into Eq. (9a), and making the coefficients of each power of r equal to zero, we obtain the following conditions on the self-similar variable and the parameter functions: $\theta(z, r) = \frac{r^2}{w^2(z)}$, $b(z) = 0$, and $a = \frac{1}{2w}\frac{dw}{dz}$. By using these conditions, a nonlinear differential equation for $F(\theta)$ is readily derived from Eq. (9b),

$$\frac{\theta}{F}\frac{\partial^2 F}{\partial\theta^2}+\frac{1}{F}\frac{\partial F}{\partial\theta}-\frac{w^3\theta}{4}\frac{d^2w}{dz^2}-\frac{w^2}{2}\frac{dc}{dz}-\frac{1}{2}sw^4\theta=0. \quad (10)$$

After another functional transformation $F(\theta)=e^{-\frac{\theta}{2}}f(\theta)$, Eq. (10) is brought to

$$\theta\frac{d^2f}{d\theta^2}+(1-\theta)\frac{df}{d\theta}+\left[\left(\frac{1}{4}-\frac{w^3}{4}\frac{d^2w}{dz^2}-\frac{1}{2}sw^4\right)\theta-\frac{w^2}{2}\frac{dc}{dz}-\frac{1}{2}\right]f=0. \quad (11)$$

It is possible to further simplify Eq. (11), with the choices:

$$\frac{1}{4}-\frac{w^3}{4}\frac{d^2w}{dz^2}-\frac{1}{2}sw^4=0, \quad (12a)$$

$$-\frac{w^2}{2}\frac{dc}{dz}-\frac{1}{2}=n, \quad (12b)$$

where n is chosen as a real nonnegative integer. These changes turn Eq. (11) into

$$\theta\frac{d^2f}{d\theta^2}+(1-\theta)\frac{df}{d\theta}+nf=0, \quad (13)$$

which is the well-known Laguerre differential equation, whose solutions are the Laguerre polynomials [29], namely $f(\theta)=L_n(\theta)$.

Here, we consider only solutions with constant width. When $w=w_0$, we find that the beam is a shape-preserving accessible soliton [16, 24, 26]. In that case, from Eqs. (12a) and (12b), it follows that the parameter functions can be deduced as $s=\frac{1}{2w_0^4}$, $c(z)=-\frac{2n+1}{w_0^2}z+c_0$, and $a=0$. Thus, a linear growth in z of the phase shift is noted, which is a usual occurrence in transversely localized beams possessing propagation constants. Collecting the above results, we get the following particular soliton solution of Eq. (1):

$$u_{nm}(z,r,\phi)=\frac{k}{w_0}L_n\left(\frac{r^2}{w_0^2}\right)\times e^{-\frac{r^2}{2w_0^2}+\cos(2m\phi)+iI_0\sin[(2m+1)\phi]+i(-\frac{2n+1}{w_0^2}z+c_0)}, \quad (14)$$

where $k=\sqrt{1/\int_0^{2\pi}e^{2\cos(2m\varphi)}d\varphi}$. It is straightforward to see that $|u(z,r,\varphi)|$ vanishes at $r\to\infty$, i.e., Eq. (14) represents a localized solitary solution. Arbitrariness in the choice of parameters n and m included in the above solution (14) implies that the beam field $u(z,r,\varphi)$ may possess a rich structure. Thus, we can conclude that to support the solution (14), PT-symmetric potential in Eq. (1) has to be in the form of Eqs. (5), where $\Omega(\varphi)$ is given by Eq. (8).

3 Analysis and discussion of results

To better understand the influence of PT-symmetric potentials on the soliton dynamics, we introduce some special types of localized solutions for the optical field expressed by Eq. (14), via suitable selections of the nonnegative integers n and m. We focus attention on the distributions of the optical intensity $I=|u|^2$ and discuss the corresponding $\Omega_R(\varphi)$ and $\Omega_I(\varphi)$ in Eq. (8). In that case, there is no z-dependence in the distributions obtained. In general, the exact expressions for $\Omega_R(\varphi)$ and $\Omega_I(\varphi)$ are likely to be of some unusual and complex form, therefore, we only provide the graphs of $\Omega_R(\varphi)$ and $\Omega_I(\varphi)$. In the following examples, we further fix the constant $w_0=1$.

We first address the case $m=0$ and different n. Because the parameter n is an arbitrary nonnegative integer, various structures are obtained. If the parameter n is chosen as zero, from Eq. (14) we find that $L_0(r^2)=1$; the beam then is called the fundamental soliton, which forms a full circle in the projection on the x–y plane. Similar excitations have appeared in our previous work [24]. An example with nonzero n is presented in Fig. 1(a) for $n=2$; the left column depicts the distribution of the intensity, while the right column displays the projection on the x–y plane. Note that the intensity is nonzero in the white rings and the central spot, and zero elsewhere. Similarly, we can construct higher order soliton excitations for larger n, i.e., for $n=4$; the distribution is exhibited in Fig. 1(b). In general, there exist n ($n\neq 0$) ring layers and a central circle for such a soliton. The maximum optical intensity is located at the center.

The real component Ω_R and the imaginary component Ω_I of the corresponding PT-symmetric potential for the parameter $m=0$ given by Eq. (8) are shown in Fig. 2. Because $\Omega(\varphi)$ contains a periodic modulation, coming from $\cos(2\varphi)$ and $\sin(\varphi)$, the PT-symmetric potential shows the periodic sine and cosine structure. Here, exceptionally, the amplitude $|\Omega_I|>|\Omega_R|$. We find that when r is small, owing to the impact of the

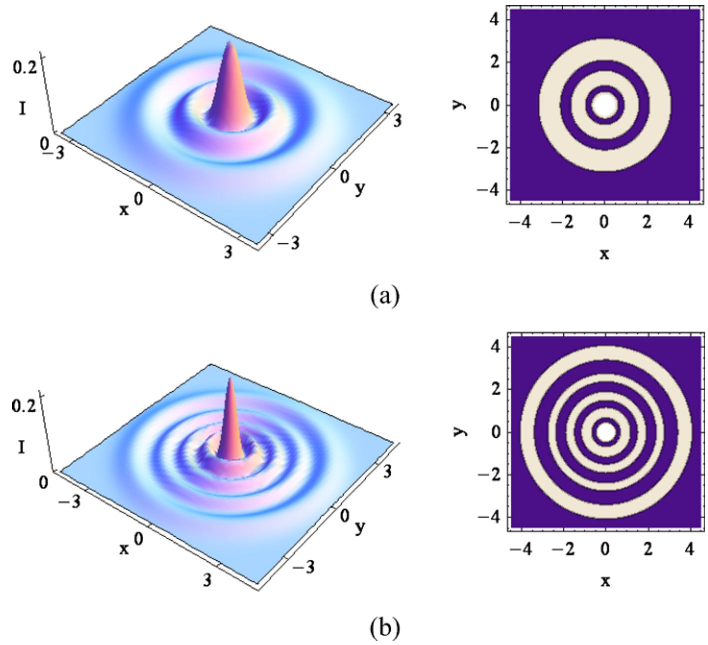

Fig. 1 Intensity distributions (nonzero in white annuli, zero in other areas) of the axially symmetric soliton structures. Top and bottom rows display $n = 2$ and $n = 4$ solitons, respectively, as given by Eq. (14) (Color figure online)

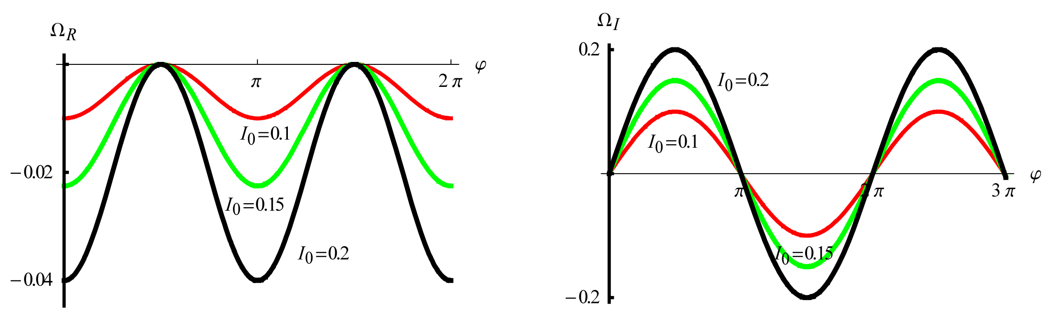

Fig. 2 The real (*left*) and imaginary (*right*) components of $\Omega(\varphi)$ from Eq. (8), displayed in the azimuthal angle for different I_0. Here $m = 0$, and $I_0 = 0.1, 0.15, 0.2$, respectively (Color figure online)

function $\frac{1}{r^2}$, the *PT*-symmetric potential is large, then decreases and oscillates rapidly, and in the end tends to zero with the increase in r.

For $n = 0$ in Eq. (14), we obtain a radiating necklace-shaped beam for any positive integer m. A typical example of such a radiating necklace is shown in Fig. 3(a) for $m = 4$, displaying an axisymmetric radial intensity distribution. The number of radiating petals in the necklace (white regions in the figure) in the soliton is determined by the integer m; there exist $2m$ petals. Actually, the formation of a radiating necklace beam is the result of the periodic azimuthal modulation function $\cos(2m\varphi)$. The corresponding real and imaginary components of $\Omega(\varphi)$ are displayed in Figs. 3(b) and 3(c). It should be noted that $\Omega_R(\varphi)$ swings violently from positive to negative values, while there are relatively small oscillations at the top (see Fig. 3(b)), whereas $\Omega_I(\varphi)$ oscillates more gently between positive and negative values with a much smaller amplitude. This wide disparity in the size of real and imaginary components helps understand the absence of phase transition in our system and the apparent stability of the solutions obtained.

Self-trapped localized structures with a large number of azimuthal petals and multilayer necklaces may exhibit a strong effective stabilization in strongly NNL media [24, 25]. Figures 4 and 5 display the intensity distribution of multilayer radiating necklace solitons and the corresponding $\Omega(\varphi)$, which exhibit similar patterns. These examples are obtained for positive in-

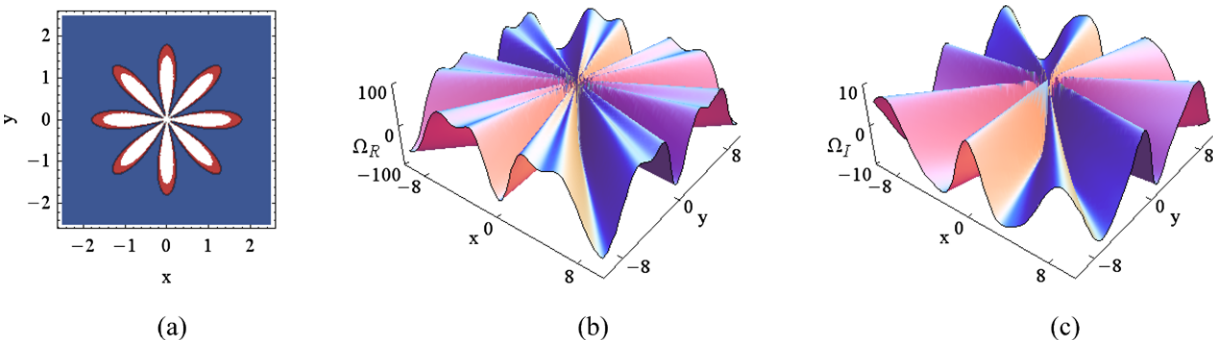

Fig. 3 (**a**) Intensity distribution of a soliton for $n = 0$ and $m = 4$; (**b**) the corresponding real component $\Omega_R(\varphi)$, and (**c**) the imaginary component $\Omega_I(\varphi)$ of the PT-symmetric potential, for $I_0 = 0.1$ (Color figure online)

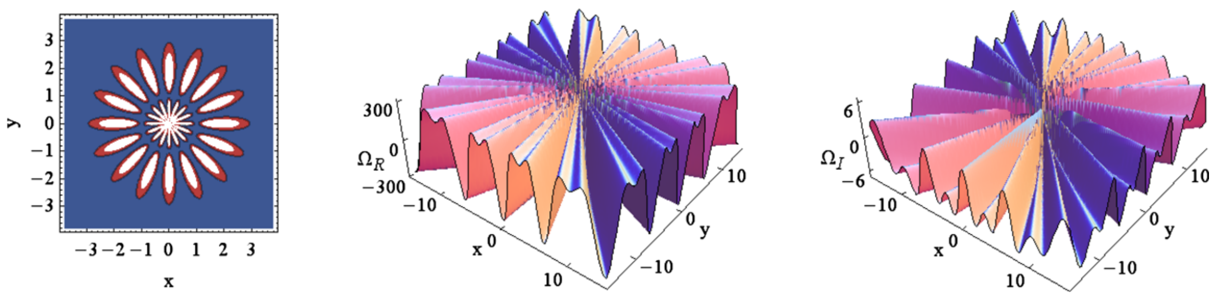

Fig. 4 Multilayer necklace soliton and the corresponding $\Omega(\varphi)$. The setup and parameters are as in Fig. 3, except for $n = 1$ and $m = 8$ (Color figure online)

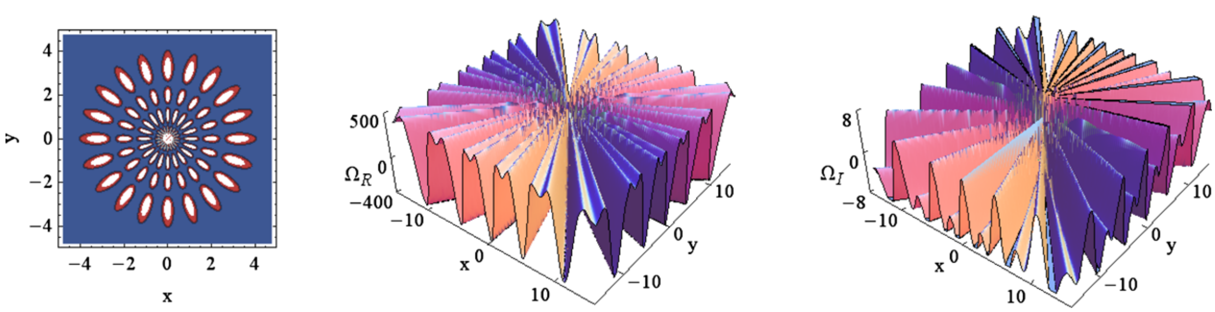

Fig. 5 Structure of multilayer necklace soliton and the corresponding $\Omega(\varphi)$. The setup is the same as in Fig. 4, except for $n = 3$ and $m = 10$ (Color figure online)

teger values of n and m in Eq. (14). In these solutions, the necklace structure is formed due to the periodic azimuthal modulation. Note that these solitons form multilayered structures, with the outer necklaces more strongly modulated than the inner counterparts; there exist larger white necklaces in the outer rings than in the inner.

Interesting structures are seen in Figs. 4 and 5. We find that the larger the parameter m, the larger the necklace radius. It is seen that the distributions change regularly with the azimuthal angle. When m is large enough, the bright spots form a soliton ring (see Fig. 1). The number of petals in each layer is determined by m, and the number of layers is determined by n. These solitons contain $2m(n + 1)$ white spots and form $n + 1$ necklace layers.

Figures 4(b), (c) and 5(b), (c) illustrate similar structures of the corresponding $\Omega(\varphi)$ as in Fig. 3. For larger m, there also appear small oscillations in $\Omega_I(\varphi)$ at the bottom (see Figs. 4(c) and 5(c)).

4 Conclusions

We have introduced a class of self-trapped beam solutions of the nonlinear Schrödinger equation in the strongly NNL media with *PT*-symmetric potentials. A specific class of *PT*-symmetric potentials is introduced, connected with the choice of the azimuthal part of the total solution to the NNLS equation, and allowing the existence of solitary solutions. Exact analytical soliton solutions are constructed by using the self-similar method and the separation of variables. The accessible solitons obtained are stable in propagation and show no tendency to collapse. These features are probably the consequence of choosing the imaginary part of the *PT*-symmetric potential much smaller than the real part (except when $m = 0$), which prevents the appearance of phase transition in the system and retains the solutions within the same class. We have considered a simple exponential form containing periodic sine and cosine functions, to describe the azimuthal angle part of the total solution. Further possible generalizations of these results are to consider 3D problems involving light bullets, and searching for alternative solutions involving different angular components. An interesting topic is also to increase the strength I_0 of the imaginary component and investigate the emergence of phase transition and instabilities in the solutions.

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References

1. Makris, K.G., Ganainy, R.E., Christodoulides, D.N., Musslimani, Z.H.: PT-symmetric optical lattices. Phys. Rev. A **81**, 063807 (2010)
2. Bendix, O., Fleischmann, R., Kottos, T., Shapiro, B.: Exponentially fragile PT symmetry in lattices with localized eigenmodes. Phys. Rev. Lett. **103**, 030402 (2009)
3. West, C.T., Kottos, T., Prosen, T.: PT-symmetric wave chaos. Phys. Rev. Lett. **104**, 054102 (2010)
4. Bender, C.M., Brody, D.C., Jones, H.F.: Complex extension of quantum mechanics. Phys. Rev. Lett. **89**, 270401 (2002)
5. Bender, C.M.: Making sense of non-Hermitian Hamiltonians. Rep. Prog. Phys. **70**, 947–1018 (2007)
6. Ruter, C.E., Makris, K.G., El-Ganainy, R., Chiristodoulides, D.N., Segev, M., Kip, D.: Observation of parity–time symmetry in optics. Nat. Phys. **6**, 192–195 (2010)
7. Lin, Z., Ramezani, H., Eichelkraut, T., Kottos, T., Cao, H., Christodoulides, D.N.: Unidirectional invisibility induced by PT-symmetric periodic structures. Phys. Rev. Lett. **106**, 213901 (2011)
8. Makris, K.G., El-Ganainy, R., Christodoulides, D.N., Musslimani, Z.H.: Beam dynamics in PT symmetric optical lattices. Phys. Rev. Lett. **100**, 103904 (2008)
9. Musslimani, Z.H., Makris, K.G., El-Ganainy, R., Christodoulides, D.N.: Optical solitons in PT periodic potentials. Phys. Rev. Lett. **100**, 030402 (2008)
10. Makris, K.G., El-Ganainy, R., Christodoulides, D.N., Musslimani, Z.H.: PT-symmetric periodic optical potentials. Int. J. Theor. Phys. **50**, 1019 (2011)
11. Abdullaev, F.Kh., Kartashov, Y.V., Konotop, V.V., Zezyulin, D.A.: Solitons in PT-symmetric nonlinear lattices. Phys. Rev. A **83**, 041805 (2011)
12. He, Y., Zhu, X., Mihalache, D., Liu, J., Chen, Z.: Lattice solitons in PT-symmetric mixed linear-nonlinear optical lattices. Phys. Rev. A **85**, 013831 (2012)
13. He, Y., Zhu, X., Mihalache, D., Liu, J., Chen, Z.: Solitons in PT-symmetric optical lattices with spatially periodic modulation of nonlinearity. Opt. Commun. **285**, 3320–3324 (2012)
14. Chen, L., Li, R., Yang, N., Chen, D., Li, L.: Optical modes in PT-symmetric double-channel waveguides. Proc. Rom. Acad., Ser. A : Math. Phys. Tech. Sci. Inf. Sci. **13**, 46–54 (2012)
15. Yin, C., He, Y., Li, H., Xie, J.: Solitons in parity-time symmetric potentials with spatially modulated nonlocal nonlinearity. Opt. Express **20**, 19355–19362 (2012)
16. Wang, M., Tian, B., Shan, W., Lü, X., Xue, Y.: Solitons and their collisions in the spinor Bose–Einstein condensates. Nonlinear Dyn. **69**, 1137–1148 (2012)
17. Snyder, A.W., Mitchell, D.J.: Accessible solitons. Science **276**, 1538–1541 (1997)
18. Mitchell, D.J., Snyder, A.W.: Dynamics in a nonlocal medium. J. Opt. Soc. Am. B **16**, 236–239 (1999)
19. Conti, C., Peccianti, M., Assanto, G.: Route to nonlocality and observation of accessible solitons. Phys. Rev. Lett. **91**, 073901 (2003)
20. Conti, C., Peccianti, M., Assanto, G.: Observation of optical spatial solitons in a highly nonlocal medium. Phys. Rev. Lett. **92**, 113902 (2004)
21. Kartashov, Y.V., Torner, L., Vysloukh, V.A., Mihalache, D.: Multipole vector solitons in nonlocal nonlinear media. Opt. Lett. **31**, 1483–1485 (2006)
22. Mihalache, D., Mazilu, D., Lederer, F., Malomed, B.A., Kartashov, Y.V., Crasovan, L.-C., Torner, L.: Three-dimensional spatiotemporal optical solitons in nonlocal nonlinear media. Phys. Rev. E **73**, 025601 (2006)
23. He, Y.J., Malomed, B.A., Mihalache, D., Wang, H.Z.: Spinning bearing-shaped solitons in strongly nonlocal nonlinear media. Phys. Rev. A **77**, 043826 (2008)
24. Zhong, W.P., Yi, L.: Two-dimensional Laguerre–Gaussian soliton family in strongly nonlocal nonlinear media. Phys. Rev. A **75**, 061801 (2007)

25. Zhong, W.P., Yi, L., Xie, R.H., Belić, M., Chen, G.: Robust three-dimensional spatial soliton clusters in strongly nonlocal nonlinear media. J. Phys. B, At. Mol. Opt. Phys. **41**, 025402 (2008)
26. Belić, M., Zhong, W.P.: Two-dimensional spatial solitons in highly nonlocal nonlinear media. Eur. Phys. J. D **53**, 97–106 (2009)
27. Zhong, W.P., Belić, M.: Three-dimensional optical vertex and necklace solitons in highly nonlocal nonlinear media. Phys. Rev. A **79**, 023804 (2009)
28. Levail, G.: Solvable PT-symmetric potential in higher dimensions. J. Phys. A, Math. Theor. **40**, 273–281 (2007)
29. Zwillinger, D.: Handbook of Differential Equations, 3rd edn. Academic Press, Boston (1997)