Photorefractive ring oscillators

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We consider the theory of photorefractive ring oscillators, using our unified solution method. Both unidirectional and bidirectional ring resonators are analyzed, based on the two-wave mixing process with crossed polarization and the four-wave mixing process with parallel polarization in photorefractive crystals. We highlight symmetries between the transmission and the reflection geometries of these processes and use them to write analytical expressions for oscillation conditions in all the cases. Symmetry breaking is noted in the four-wave mixing between the transmission and the reflection grating cases. An optical transistor based on photorefractive rings is proposed.

1. INTRODUCTION

The ring resonator is one of the paradigms in nonlinear optics.[1] It is a popular choice for a unidirectional laser cavity. It is the geometry of choice for investigations of optical bistability. It is used in laser gyros, phase-conjugate mirrors, light switches, interferometers, and other optical and optoelectronic devices.

We consider a ring with a piece of photorefractive (PR) crystal inserted as an active or passive optical element. Such a device shows great potential for applications in PR nonlinear optics.[2] Various geometries of interest are presented in Fig. 1. Figure 1(a) represents a unidirectional ring based on vectorial two-wave mixing (2WM) in PR media in transmission geometry (TG) when all fields are incident upon the same face of the crystal; Fig. 1(b) depicts a unidirectional ring in reflection geometry (RG) when the fields are incident upon the opposite faces; Fig. 1(c) represents the bidirectional ring based on four-wave mixing (4WM), which can be TG or RG, depending on the manner in which the interacting fields build diffraction gratings. We discuss oscillation conditions for all these geometries. We highlight symmetries among different geometries and different processes. We propose an all-optical device based on PR rings that operates as an optical transistor.

The most comprehensive analysis of PR rings to date was made by Yeh.[2,3] However, his analysis was restricted to 2WM unidirectional rings and 4WM bidirectional rings with parallel polarizations and transmission gratings. We generalize the treatment by including the vectorial (cross-polarization) coupling and reflection gratings. The solution method is also a generalization of the method used by Gu and Yeh[3] that we[4] recently geared for a more comprehensive treatment of different wave-mixing processes.

There are two sides to the problem of PR rings. The first one is the solution of wave-mixing equations in the PR crystal subject to appropriate boundary conditions. The second is the analysis of oscillation conditions. The method of solution is based on the fact that, when the diffusion of photoinduced charge carriers is responsible for the buildup of diffraction gratings, the coupling constants of wave mixing are real and the phase of the grating in question is constant.[5] This statement applies equally well to vectorial and scalar coupling and to TG and RG. The oscillation conditions imply the application of conditions for resonator mode oscillation.

There is an aspect common to both sides of the problem: the question of phase shifts and frequency detunings in the ring. It is known that the PR effect may induce a phase shift in each of the mixing beams. These shifts can be offset by cavity tuning. Conversely, cavity detuning can be compensated by a PR phase shift. This supposedly happens in some PR oscillators that spontaneously detune from the pump frequency. However, such shifts make the analysis complicated. In addition, compensation schemes might not work in bidirectional rings, in which clockwise and counterclockwise optical paths might not be the same (owing to nonreciprocity). This question apparently has not been settled in the literature.[6] To make the analysis tractable we assume that the mechanism for charge carrier redistribution is only diffusion, in which case there is no PR detuning, and that the cavity is tuned. For the 4WM bidirectional ring we assume in addition that the optical paths are reciprocal.

Wave-mixing processes in PR crystals are governed by the slowly varying envelope wave equations. For vectorial 2WM and scalar 4WM they are of the forms

$$IA_s' = \sigma\Gamma Q B_p, \qquad IA_p' = \sigma\Gamma Q B_s\,, \tag{1a}$$

$$1B_s' = \Gamma\overline{Q} A_p, \qquad IB_p' = \Gamma\overline{Q} A_s\,, \tag{1b}$$

$$IA_1' = \sigma\Gamma Q A_3, \qquad IA_3' = \Gamma\overline{Q} A_1\,, \tag{2a}$$

$$IA_2' = \sigma\Gamma\overline{Q} A_4, \qquad IA_4' = \Gamma Q A_2\,, \tag{2b}$$

where A_s, A_p, B_s, B_p and A_1, A_2, A_3, A_4 denote the waves

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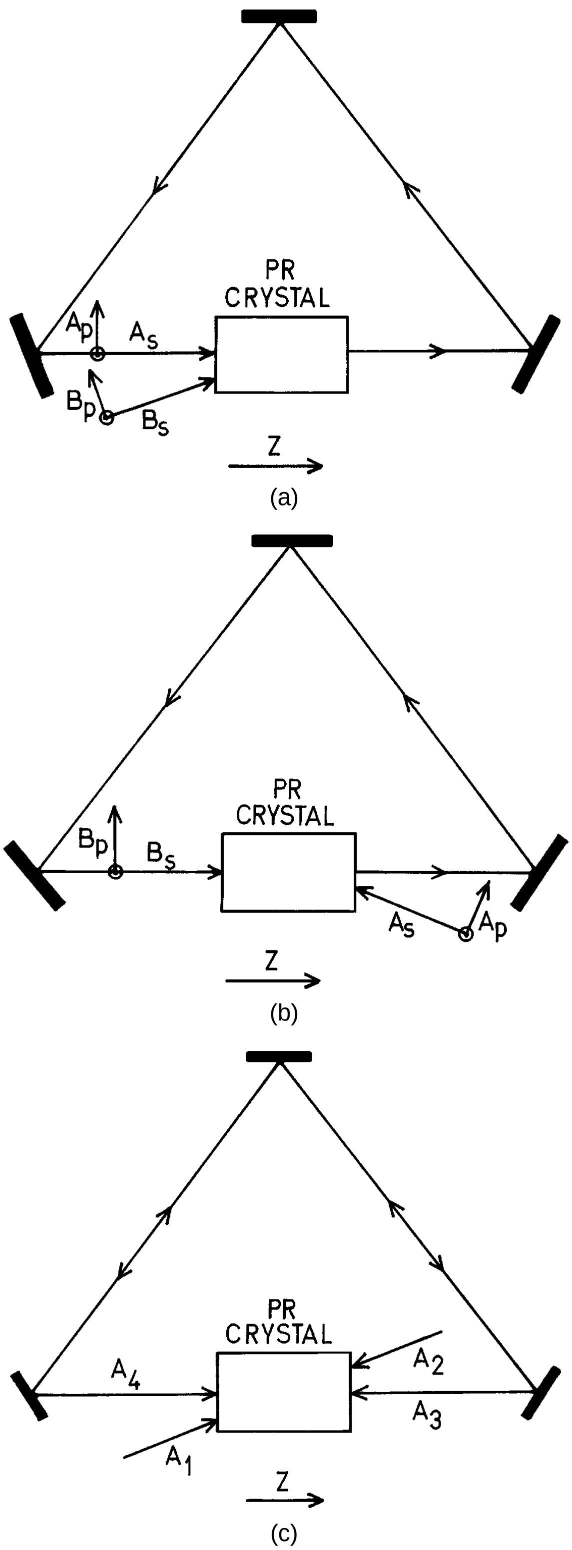

Fig. 1. Photorefractive ring resonators considered: (a) 2WM unidirectional ring with cross coupling in TG, (b) 2WM unidirectional ring in RG, (c) 4WM bidirectional ring with parallel coupling in either TG or RG.

that are mixed in the crystal and $\sigma = \pm$ is the parameter controlling the geometry. $\sigma = +$ corresponds to RG, and $\sigma = -$ corresponds to TG. I is the total intensity ($I_A + I_B$ in 2WM, $I_1 + I_2 + I_3 + I_4$ in 4WM), Q is the amplitude of the grating ($A_s\overline{B}_s + A_p\overline{B}_p$ in 2WM, $A_1\overline{A}_3 + \overline{A}_2 A_4$ in 4WM). Γ is the wave-coupling constant, given in inverse centimeters. A prime denotes a derivative in the propagation (z) direction across the crystal, and a bar stands for complex conjugation. A steady-state, plane-wave, degenerate situation is assumed. We should note that Eqs. (2) for 4WM as written are good only for the RG. To obtain the TG equations, in addition to taking $\sigma = -$ one should exchange the fields $A_3 \rightleftharpoons A_4$. We will keep the RG notation, as we are more interested in the RG case (TG has been adequately discussed by Gu and Yeh[3]). However, when we want to complete the Gu–Yeh analysis, or to draw one's attention to the symmetries and the symmetry breaking between TG and RG, we will use both kinds of notation.

The notation could be made even more economical, because the method of solution for both 2WM and 4WM is essentially the same. Nevertheless, we prefer to keep the treatment of 2WM and of 4WM separate. Hence in Section 2 we present an analysis of the cross-polarization 2WM ring oscillator, an analysis that to our knowledge was not performed before. In Section 3 we perform a similar analysis of the scalar 4WM ring in the RG. We point out where and why an analogous Gu–Yeh analysis of the TG ring in contrast had to be performed numerically. In Section 4 our results are discussed and conclusions are drawn.

2. TWO-WAVE MIXING RINGS

The solution procedure has been described at length in our other publications.[4,5] Here we present only an outline inasmuch as it is needed for the discussion of the ring oscillators. As mentioned, the procedure is based on the fact that the phase ϕ of the grating amplitude is constant for diffusion-dominated buildup of gratings. In that case the phase shift between the light interference pattern and the grating is $\pi/2$, and the coupling constant Γ is a real number. There exists a convenient independent variable θ, defined by

$$\theta' = \Gamma |Q|/I\,, \tag{3}$$

in terms of which the original equations become linear. The solution of Eqs. (1) is given by

$$A_s = A_{s0}c(\Theta) + \sigma B_{p0}\exp(i\phi)s(\Theta)\,, \tag{4a}$$

$$B_p = B_{p0}c(\Theta) + A_{s0}\exp(-i\phi)s(\Theta)\,, \tag{4b}$$

$$A_p = A_{p0}c(\Theta) + \sigma B_{s0}\exp(i\phi)s(\Theta)\,, \tag{4c}$$

$$B_s = B_{s0}c(\Theta) + A_{p0}\exp(-i\phi)s(\Theta)\,, \tag{4d}$$

where the functions $s(\Theta)$ and $c(\Theta)$, respectively, denote the hyperbolic sine and cosine functions for $\sigma = +$ and the cyclic sine and cosine functions for $\sigma = -$ and Θ is an abbreviation for $\theta - \theta_0$. Again, ϕ is the constant phase of the grating (which can be set to zero). The solution is given in terms of the initial values taken at $z = 0$. Whereas this is natural in TG, in RG one must reexpress the initial values in terms of the given two-point boundary conditions:

$$A_{s0} = A_{sd}\,\mathrm{sech}(u) - B_{p0}\tanh(u)\exp(i\phi)\,, \tag{5a}$$

$$A_{p0} = A_{pd}\,\mathrm{sech}(u) - B_{s0}\tanh(u)\exp(i\phi)\,, \tag{5b}$$

where $u = \theta_d - \theta_0$ and d denotes the thickness of the crystal. The quantity u is the grating action. It is important in the overall discussion. In fact, to every crystal one can assign its own grating action, which depends on the beam-coupling strength and on the wave-mixing geometry. As a rule, the grating action must be determined numerically. The complete solution also requires the knowledge of θ_0 as a function of the boundary conditions and of θ as a function of z. However, in our analysis we need only the values of the fields at the entrance and the exit faces of the crystal.

We determine the function $\Theta(z)$ by inserting the expressions for the fields into Eq. (3) and evaluating the quadrature. The results for TG and RG are different, owing to the fact that the total intensity I is constant in TG whereas it is not in RG:

$$\text{TG:}\quad \tan(\Theta) = \frac{|Q_0|}{\delta \coth\left(\dfrac{\delta\Gamma z}{I_0}\right) - P_0}, \tag{6a}$$

$$\text{RG:}\quad \Gamma z = \frac{I_0 - J_0}{|Q_0| - P_0}\,\Theta + \frac{J_0|Q_0| - I_0 P_0}{2(|Q_0|^2 - P_0{}^2)} \times \ln\left|\frac{(|Q_0| + P_0)\exp(4\Theta) + |Q_0| - P_0}{2|Q_0|}\right|, \tag{6b}$$

where $2P = A_s\overline{A}_p + \sigma B_s\overline{B}_p + \text{c.c.}$, $\delta = (|Q_0|^2 + P_0{}^2)^{1/2}$, and $J = (A_s\overline{B}_p + A_p\overline{B}_s)\exp(-i\phi) + \text{c.c.}$ On the solutions given by Eqs. (4) one must impose cavity oscillation conditions. As in any other cavity, the mode oscillation condition is that the intracavity field reproduces itself after a round trip. This puts an extra condition on the two field components that were considered free parameters above.

In TG the intracavity field is the A field, so the oscillation conditions read as

$$A_{p0} = r\exp(i\psi)A_{pd}, \qquad A_{s0} = r\exp(i\psi)A_{sd}, \tag{7}$$

where r is the product of the reflectivities of the three mirrors that constitute the ring and ψ is the round-trip propagation phase shift. We assume that $|r|$ can be larger than 1, so the presence of optical amplifiers in the ring is allowed. Assuming that the overall reflectivity produces a phase shift ϕ_r, the round-trip phase condition is given by

$$\psi + \phi_r = 2m\pi, \tag{8}$$

where m is an integer. The ring will oscillate, provided that condition (8) is fulfilled and the intensities $|A_{p0}|^2$ and $|A_{s0}|^2$ are nonzero. There might also be threshold conditions forced on the other free parameters in the problem: the pump-field intensity $|B|^2$ and the strength of the coupling Γd. We discuss these below.

Usually there is also a frequency condition on the oscillating cavity. However, the cavity containing a PR crystal can oscillate at any cavity length, even though the bandwidth is narrow (provided that the coupling is strong enough).[3] For simplicity we assume degeneracy (no frequency detunings) and a tuned cavity. With the help of Eqs. (4) and (7) we obtain the amplification coefficient:

$$\kappa = \frac{A_{s0}}{B_{p0}}\exp(-i\phi) = \frac{A_{p0}}{B_{s0}}\exp(-i\phi) = \frac{\sin(u)}{\cos(u) - \dfrac{1}{|r|}}\cdot \tag{9}$$

Using Eqs. (6a) and (9), we find the following expression for the grating action:

$$\tan(u) = \frac{\kappa(b - c)}{b + \kappa^2 c}, \tag{10}$$

where $b = \coth(\delta\Gamma d/I_0) + 1$ and $c = \coth(\delta\Gamma d/I_0) - 1$. From Eqs. (9) and (10) it follows that

$$\cos(u) = \frac{b + c|r|^2}{|r|(b + c)}, \tag{11}$$

and this expression can be used to find the threshold conditions on the coupling strength Γd. After a short algebraic manipulation, one obtains the conditions that are put together in Table 1. We define $\tau = b/c = \exp(2\delta\Gamma d/I_0) = \exp(a\Gamma d)$ as the threshold parameter. In RG (to be discussed in a moment) the threshold parameter will be $\exp(-a\Gamma d)$. Table 1 applies to both TG and RG. The threshold value of the coupling, Γ_{th}, is determined from the condition that $\tau = |r|$ at the threshold. One obtains

$$\Gamma_{\text{th}} d = \ln(|r|)/a, \tag{12}$$

where $a = 2\delta/I_0 = (B_{s0}\overline{B}_{p0} + \text{c.c.})/(|B_{s0}|^2 + |B_{p0}|^2)$. We anticipate that the threshold coupling for RG will be exactly the negative of the one for TG, as shown in Figs. 2 and 3.

Table 1. Threshold Conditions

2WM	TG, RG
$1 < \|r\|$	$\|r\| < \tau$
$\|r\| < 1$	$\tau < \|r\|$

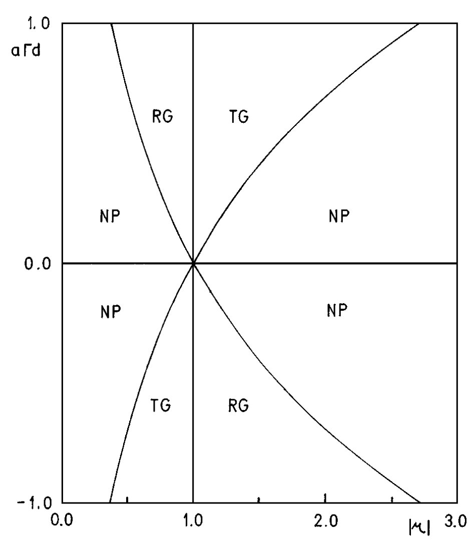

Fig. 2. Threshold conditions for 2WM rings from Table 1 presented graphically: NP, not possible. For these values of overall reflectivity of the cavity $|r|$ and coupling strength $a\Gamma d$ the ring will not oscillate.

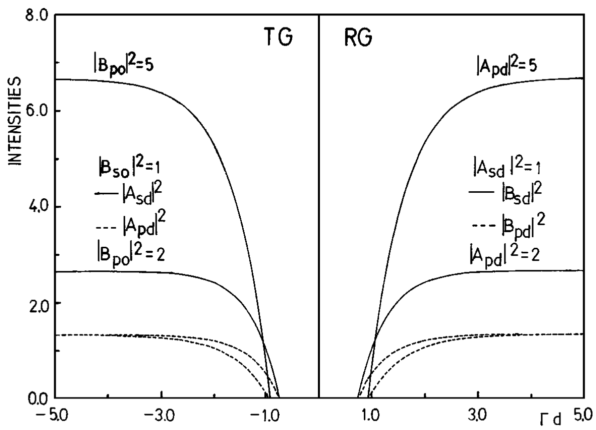

Fig. 3. Oscillation intensities $|A_{sd}|^2$ and $|A_{pd}|^2$ (TG side) and $|B_{sd}|^2$ and $|B_{pd}|^2$ (RG side) in the 2WM ring as functions of coupling Γd for two values of the pump intensity ratio. Perfect symmetry between TG and RG is visible. Here $|r| = 0.5$ and $\alpha = 0$.

Using these results, one finds convenient expressions for the oscillating intracavity intensities:

$$|A_{pd}|^2 = \frac{|B_{s0}|^2}{1 - |r|^2}\left(1 - \frac{\tau^2}{|r|^2}\right), \tag{13a}$$

$$|A_{sd}|^2 = \frac{|B_{p0}|^2}{1 - |r|^2}\left(1 - \frac{\tau^2}{|r|^2}\right). \tag{13b}$$

These formulas are obtained from Eqs. (7) and (9). Figure 3 depicts some representative cases. Thus the intracavity field can be amplified (or changed in other ways) at the expense of pumps. We use these formulas below to display transistor action in PR rings.

In the RG the intracavity field is the B field, so the oscillation condition has the form

$$B_{p0} = r\exp(i\psi)B_{pd}, \qquad B_{s0} = r\exp(i\psi)B_{sd}. \tag{14}$$

The equivalent of Eq. (9) is then

$$\kappa = -\frac{B_{p0}}{A_{sd}}\exp(i\phi) = -\frac{B_{s0}}{A_{pd}}\exp(i\phi), \tag{15a}$$

where now

$$\kappa = \frac{\sinh(u)}{1 - \dfrac{\cosh(u)}{|r|}}. \tag{15b}$$

The grating action u is found from Eq. (6b), evaluated at $z = d$:

$$\Gamma d = \frac{I_0 - J_0}{|Q_0| - P_0}u + \frac{J_0|Q_0| - I_0P_0}{2(|Q_0|^2 - P_0{}^2)} \times \ln\left|\frac{(|Q_0| + P_0)\exp(4u) + |Q_0| - P_0}{2|Q_0|}\right|, \tag{16}$$

where $|Q_0|$, P_0, I_0, and J_0 depend only on u:

$$|Q_0| = -a\kappa I_A\,\mathrm{sech}(u)[1 + \kappa\sinh(u)], \tag{17a}$$

$$2P_0 = aI_A\,\mathrm{sech}^2(u)[1 + 2\kappa\sinh(u) + \kappa^2\cosh(2u)], \tag{17b}$$

$$I_0 = I_A\,\mathrm{sech}^2(u)[1 + 2\kappa\sinh(u) + \kappa^2\cosh(2u)], \tag{17c}$$

$$J_0 = -2\kappa I_A\,\mathrm{sech}(u)[1 + \kappa\sinh(u)], \tag{17d}$$

where now $a = (A_{sd}\bar{A}_{pd} + \text{c.c.})/(|A_{sd}|^2 + |A_{pd}|^2)$ and $I_A = |A_{sd}|^2 + |A_{pd}|^2$. The oscillation condition imposes a relation on the quantities $|Q_0|$, P_0, I_0, and J_0:

$$I_0|Q_0| = J_0P_0, \tag{18}$$

which together with Eqs. (17) is used to simplify Eq. (16):

$$a\Gamma d = \ln\left|\frac{|Q_0|\cosh(2u) + P_0\sinh(2u)}{|Q_0|}\right|. \tag{19}$$

This brings an expression for the grating action analogous to Eq. (10):

$$\sinh(u) = \frac{\kappa(b - c)}{b + \kappa^2 c}. \tag{20}$$

Here $b = 1$ and $c = \exp(a\Gamma d)$, so $\tau = \exp(-a\Gamma d)$. In TG, remember that $\tau = \exp(a\Gamma d)$. Thus the threshold coupling for RG is the negative of the threshold coupling for TG. In view of Eq. (15b), it follows that

$$\cosh(u) = \frac{|r|(b + c)}{b + c|r|^2}. \tag{21}$$

Because $\cosh(u) \geq 1$, the threshold conditions from Table 1 for RG are obtained. Likewise, the expressions for the oscillation fields analogous to Eqs. (13) follow.

One can see that a high degree of symmetry exists between TG and RG in the 2WM ring with crossed polarization. We will see that this symmetry is formally carried over to the 4WM rings. However, at one point it will be broken. Our results for PR rings are summarized in Table 2. 4WM rings are discussed in Section 3.

Table 2. Summary of Results

2WM, Cross Polarization		4WM, Parallel Polarization	
RG	TG	RG	TG
$\tau = \exp(-a\Gamma d)$	$\tau = \exp(a\Gamma d)$	$\tau = \dfrac{I_{2d} + I_{10}\exp(\Gamma d)}{I_{10} + I_{2d}\exp(\Gamma d)}$	$\tau \approx \dfrac{I_{10} + I_{2d}\exp(\Gamma d)}{I_{2d} + I_{10}\exp(\Gamma d)}$
$\cosh(u) = \dfrac{\lvert r\rvert(\tau + 1)}{\tau + \lvert r\rvert^2}$	$\sec(u) = \dfrac{\lvert r\rvert(\tau + 1)}{\tau + \lvert r\rvert^2}$	$\cosh(u) = \dfrac{\lvert r\rvert(\tau + 1)}{\tau + \lvert r\rvert^2}$	$\sec(u) \approx \dfrac{\lvert r\rvert(\tau + 1)}{\tau + \lvert r\rvert^2}$
$\kappa = \dfrac{\lvert r\rvert\sinh(u)}{\lvert r\rvert - \cosh(u)}$	$\kappa = \dfrac{\lvert r\rvert\sin(u)}{\lvert r\rvert\cosh(u) - 1}$	$\kappa = \dfrac{\lvert r\rvert\sinh(u)}{\lvert r\rvert - \cosh(u)}$	$\kappa = \dfrac{\lvert r\rvert\sin(u)}{\lvert r\rvert\cos(u) - 1}$
$\sinh(u) = \dfrac{\kappa(\tau - 1)}{\tau + \kappa^2}$	$\tan(u) = \dfrac{\kappa(\tau - 1)}{\tau + \kappa^2}$	$\sinh(u) = \dfrac{\kappa(\tau - 1)}{\tau + \kappa^2}$	$\tan(u) \approx \dfrac{\kappa(\tau - 1)}{\tau + \kappa^2}$

3. FOUR-WAVE MIXING RINGS

The analysis of the 4WM rings proceeds accordingly. We follow the RG case. The solution of Eqs. (2) is of the form

$$A_1 = A_{10} \cosh(\Theta) + A_{30} \exp(i\phi)\sinh(\Theta), \tag{22a}$$

$$A_3 = A_{30} \cosh(\Theta) + A_{10} \exp(-i\phi)\sinh(\Theta), \tag{22b}$$

$$A_2 = A_{20} \cosh(\Theta) + A_{40} \exp(-i\phi)\sinh(\Theta), \tag{22c}$$

$$A_4 = A_{40} \cosh(\Theta) + A_{20} \exp(i\phi)\sinh(\Theta). \tag{22d}$$

Again, the missing end values of fields are given in terms of the known fields and the grating action:

$$A_{30} = A_{3d} \operatorname{sech}(u) - A_{10} \exp(-i\phi)\tanh(u), \tag{23a}$$

$$A_{20} = A_{2d} \operatorname{sech}(u) - A_{40} \exp(-i\phi)\tanh(u). \tag{23b}$$

On inspecting Eqs. (4) and (5) one notes a complete symmetry with the 2WM RG case. However, the 4WM ring is bidirectional. The counterpropagating intracavity fields are A_4 and A_3. Hence the oscillation conditions are

$$A_{40} = r \exp(i\psi)A_{4d}, \qquad A_{3d} = r \exp(i\psi)A_{30}, \tag{24}$$

and the amplification coefficient

$$\kappa = \frac{A_{3d}}{A_{10}} \exp(i\phi) = -\frac{A_{40}}{A_{2d}} \exp(-i\phi) \tag{25}$$

has exactly the same form as in Eq. (15b). Furthermore, one can show that the quantity

$$R = I^2 - 4|Q|^2 \tag{26}$$

is constant. Hence $2|Q| = \sqrt{R} \sinh(2\theta)$, $I = \sqrt{R} \cosh(2\theta)$, and the variable θ is given by

$$\sinh(2\theta) = \sinh(2\theta_0)\exp(\Gamma z). \tag{27}$$

On the other hand, from the expressions for $|Q|$ at $z = 0$ and $z = d$ one finds that[4]

$$\sinh(u) = \frac{[\exp(\Gamma d) - 1]\,|A_{10}\overline{A}_{3d} + \overline{A}_{2d}A_{40}|}{\exp(\Gamma d)\,(I_{10} + I_{40}) + I_{2d} + I_{3d}}, \tag{28}$$

which in view of Eq. (25) is easily manipulated into the form given by Eq. (20), with $b = I_{2d} + I_{10} \exp(\Gamma d)$ and $c = I_{10} + I_{2d} \exp(\Gamma d)$. Likewise, Eq. (21) follows. In this manner column 3 of Table 2 is completed. Similarly, one can show that Table 1 is valid for the 4WM ring, provided that $\tau = b/c$ is used as the threshold parameter. However, for reasons to be discussed in a moment, Tables 1 and 2 are only approximately correct for the 4WM TG.

A high degree of symmetry is noted between the 2WM RG and the 4WM RG. One is tempted to use the apparent hyperbolic–trigonometric symmetry and the inverse threshold conditions between the 2WM RG and TG to complete Table 2 for the 4WM TG case. This is even more tempting in view of the fact that Gu and Yeh could not derive the corresponding formulas by using a similar method. Instead, they had to resort to numerical computations. However, such a program could be carried out only up to a point. At a certain place in the procedure the symmetry between the RG and the TG cases breaks, and the expressions are only approximately correct (i.e., symmetric). In what follows we explore in more detail this example of symmetry breaking.

The analysis of the 4WM TG ring proceeds along the same lines as before. We write the solution of the wave equations:

$$A_1 = A_{10} \cos(\Theta) - A_{40} \exp(i\phi)\sin(\Theta), \tag{29a}$$

$$A_4 = A_{40} \cos(\Theta) - A_{10} \exp(-i\phi)\sin(\Theta), \tag{29b}$$

$$A_2 = A_{20} \cos(\Theta) - A_{30} \exp(-i\phi)\sin(\Theta), \tag{29c}$$

$$A_3 = A_{30} \cos(\Theta) + A_{20} \exp(i\phi)\sin(\Theta), \tag{29d}$$

and we find the missing boundary values:

$$A_{20} = A_{2d} \cos(u) + A_{3d} \exp(-i\phi)\sin(u), \tag{30a}$$

$$A_{30} = A_{3d} \cos(u) - A_{2d} \exp(i\phi)\sin(u). \tag{30b}$$

The oscillation conditions read as

$$A_{40} = r \exp(i\psi)A_{4d}, \qquad A_{3d} = r \exp(i\psi)A_{30}, \tag{31}$$

and the amplification coefficient is

$$\kappa = -\frac{A_{40}}{A_{10}} \exp(i\phi) = \frac{A_{3d}}{A_{2d}} \exp(-i\phi) = \frac{\sin(u)}{\cos(u) - \dfrac{1}{|r|}}. \tag{32}$$

Furthermore, in analogy with Eq. (26), one finds that the quantity

$$T = F^2 + 4|Q|^2, \tag{33}$$

where $F = I_1 - I_3 + I_2 - I_4$ is the Poynting power flow, is constant. T is the quantity corresponding to $4\delta^2$ in the 2WM case. Thus $2|Q| = \sqrt{T} \sin(2\theta)$, $F = \sqrt{T} \cos(2\theta)$, and Eq. (3) for θ is integrated:

$$\tan(\theta) = \tan(\theta_0)\exp\left(\frac{\sqrt{T}\,\Gamma z}{I_0}\right). \tag{34}$$

The symmetry between 2WM and 4WM fails when one tries to impose oscillation conditions on the boundary values. That is,[4] to determine u one has to solve an algebraic equation for $\sqrt{T}$ and then substitute it into an appropriate formula for u:

$$\tan(u) = \frac{q}{\sqrt{T} \coth\left(\dfrac{\sqrt{T}\,\Gamma d}{2I_0}\right) - v}, \tag{35}$$

where $q = (A_{10}\overline{A}_{40} + \overline{A}_{2d}A_{3d})\exp(-i\phi) + \text{c.c.}$, $v = I_{10} - I_{2d} + I_{3d} - I_{40}$. This expression is analogous to the corresponding expression for 2WM, Eq. (6a). However, on inclusion of the oscillation conditions one is able to bring the starting expression for the 2WM grating action,

$$\tanh(u) = \frac{2\kappa(B_{s0}\overline{B}_{p0} + \text{c.c.})}{2\delta \coth\left(\dfrac{\delta \Gamma d}{I_0}\right) - (\kappa^2 - 1)(B_{s0}\overline{B}_{p0} + \text{c.c.})}, \tag{36a}$$

to the form given by Eq. (10). The corresponding expression for the 4WM case,

$$\tanh(u) = \frac{2\kappa(I_{2d} - I_{10})}{\sqrt{T}\coth\left(\dfrac{\sqrt{T}\,\Gamma d}{2I_0}\right) - (\kappa^2 - 1)(I_{2d} - I_{10})}, \tag{36b}$$

cannot be transformed into the form given by Eq. (10) because the boundary values from both faces of the crystal are mixed in. The term $I_{2d} - I_{10}$ in Eq. (36b) does not cancel out. Complete symmetry would be restored if instead of A_{20} and A_{30} in the expression for $\sqrt{T}$ in Eq. (35) one used A_{2d} and A_{3d}. This is how column 4 of Table 2 is completed. Nevertheless, an approximate symmetry is maintained, because the forms and the numerical values of both expressions remain close, as can be seen from Fig. 4. This is also visible in Fig. 5, where both the symmetric and the numerical branches are shown. One can see that the threshold condition is the same for both branches.

The threshold value of the coupling Γ_{th} is again determined from the condition $\tau = |r|$. The result is

$$\exp(\Gamma_{\text{th}} d) = \frac{p - |r|}{p|r| - 1}, \tag{37}$$

where p is the pump ratio, $p = I_{10}/I_{2d}$. One obtains the same result either from the symmetric solution in Table 2 or by making a small u expansion in Eq. (36b). By symmetry, or by a small u expansion in Eq. (28), the threshold value for the RG case is found to be $-\Gamma_{\text{th}}$. An analysis of Eq. (37) will impose some conditions on the pump ratio. For the ring to oscillate when $\Gamma > 0$ (i.e., $u > 0$) the ratio will be

$$p < \frac{1}{|r|} \quad \text{for } |r| > 1, \qquad p > \frac{1}{|r|} \quad \text{for } |r| < 1, \tag{38a}$$

and when $\Gamma < 0$ it is

$$p > |r| \quad \text{for } |r| > 1, \qquad p < |r| \quad \text{for } |r| < 1. \tag{38b}$$

The same conditions hold for the 4WM RG, provided $\Gamma \rightarrow -\Gamma$.

Another symmetry-breaking instance is the appearance of the multivalued solutions in the 4WM TG case for small values of the pump ratio [below the threshold defined by relations. (38)]. This aspect is missing from the 4WM RG case as well as from the 2WM case. Similar tendencies are noted in the behavior of the grating action u as a function of Γd (Fig. 6). This puts the 4WM TG case apart from the other cases of 2WM and 4WM rings. One can obtain the maximum value of the pump ratio for which multiple solutions still exist from Eq. (37) by putting $\Gamma_{\text{th}} = \pm\infty$. Thus for $\Gamma > 0$ and $|r| < 1$ it is $p_{\text{th}} = 1/|r|$, and for $\Gamma < 0$ and $|r| > 1$ it is $p_{\text{th}} = |r|$. This is in agreement with the numerical findings of Gu and Yeh. For values of the pump ratio for which multiple solutions exist in TG there are no solutions in RG. The value of the threshold coupling constant in the multivalued case cannot be obtained by a small u expansion (because u is not small anymore). It must be determined by other means.

One can resolve the multivaluedness by performing a stability analysis. Based on such an analysis, Wang and Pan[7] conclude that the lower branches in Figs. 5 and 6 are unstable. Based on symmetry and physical arguments we conclude that the upper branch of the multiple solution in Fig. 4 and hence the corresponding lower branches in Figs. 5 and 6 are unstable. The total grating action u should be an increasing function of the coupling strength Γd. In the end it should be mentioned that the ring oscillation in the multivalued case is not the standard PR oscillation, in which the intracavity field rises from

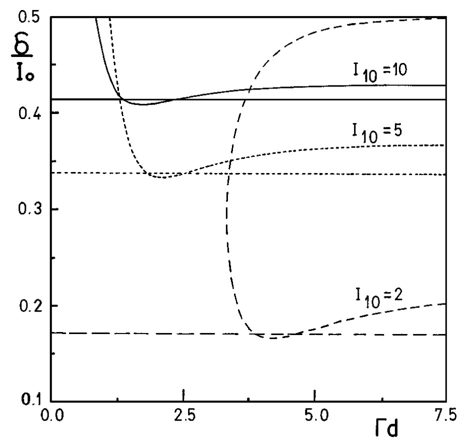

Fig. 4. Integration constant $\delta/I_0 = \sqrt{T}/2I_0$ in the 4WM TG case as a function of the coupling coefficient Γd for three values of the pump ratio I_{10}/I_{2d}. If the symmetry between TG and RG were to hold, this constant would be the same for any value of Γd (depicted by the corresponding horizontal lines). As it can be seen, the symmetry holds only approximately, and better so for higher values of the pump ratio. Other parameters are $|r| = 0.5$ and $I_{2d} = 1$.

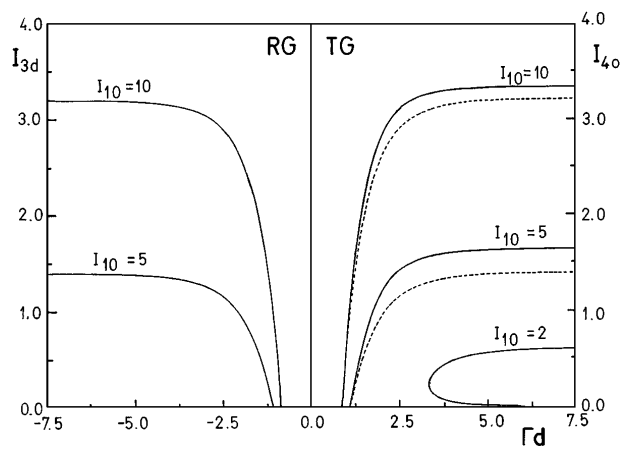

Fig. 5. Oscillation intensities I_{3d} for the RG case and I_{40} for the TG case of the 4WM bidirectional ring. This figure should be compared with Fig. 3. The dashed curves on the TG side display the symmetric approximate solution obtained by analogy with the 2WM ring. A further breakup of symmetry is noted for small values of the pump ratio I_{10}/I_{2d}: whereas in the RG there is no solution in the TG one obtains multiple solutions. The parameters are as in Fig. 4.

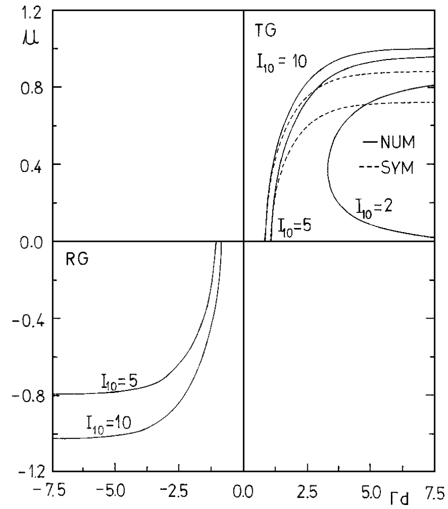

Fig. 6. Grating action u as a function of the coupling constant Γd for different values of the pump ratio in the 4WM case. The dashed curves depict the approximate symmetric solution. The parameters are as in Fig. 4. Again, multistability is noted.

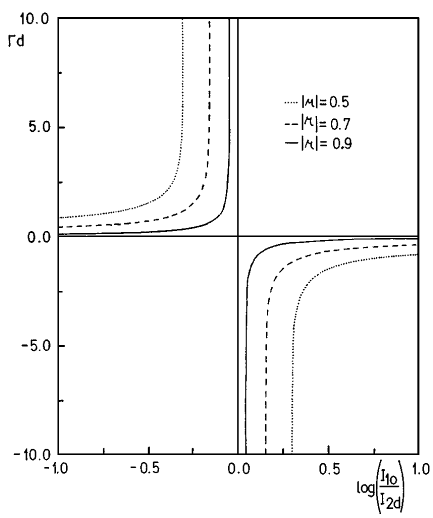

Fig. 7. Threshold value of $\Gamma_{\mathrm{th}} d$ as a function of the pump ratio for different values of $|r|$. RG is assumed. (Results for TG are identical to those of Gu and Yeh.)

the noise. Here, in addition to the coupling's being above the threshold and the pump beam ratio's being below the multiplicity threshold, a finite seeding of the intracavity field is necessary.

4. SUMMARY

Unidirectional and bidirectional ring resonators with photorefractive crystals as intracavity optical elements have been discussed in this paper. Intracavity oscillation is based on 2WM and 4WM processes in PR crystals, whereby one beam oscillates and the other acts as a pump (unidirectional ring) or two beams oscillate and two act as pumps (bidirectional ring). Analytical expressions for threshold conditions and oscillation intensities are obtained. Our results are summarized in Tables 1 and 2 and are presented graphically in Figs. 2–8.

Our results are a generalization of the results of Gu and Yeh, in that we considered both TG and RG and both

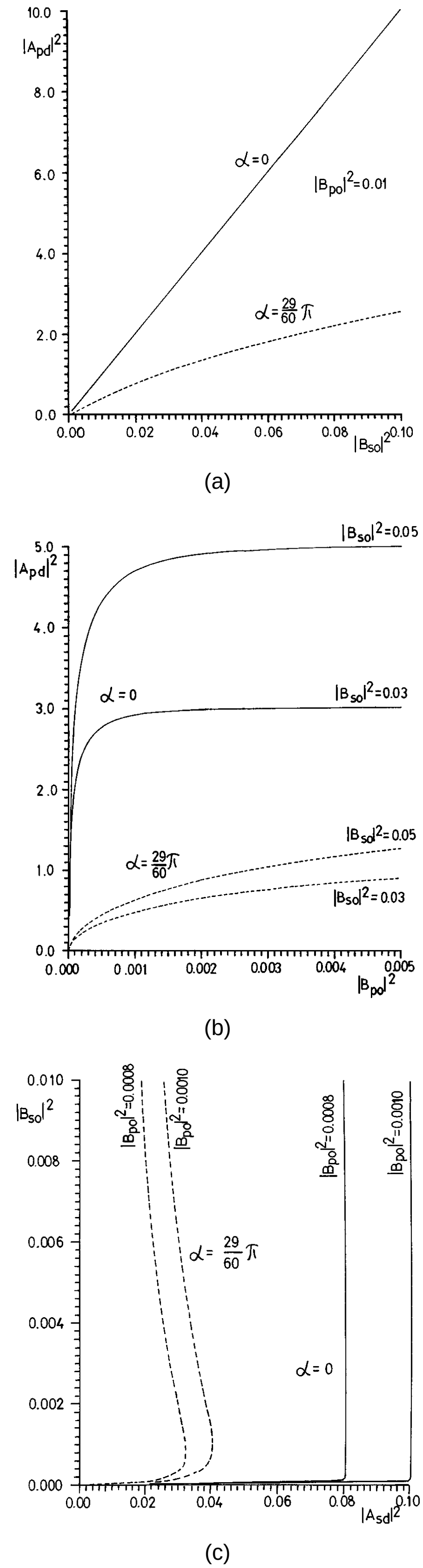

Fig. 8. Transistor action of a 2WM TG ring: (a) The amplification effect. Here $\beta = 100.2$ for $|r| = 0.995$ and $\Gamma d = -5$. The parameter α is the phase difference between the pump components. (b) The saturation effect. (c) Volt–ampere characteristics of an optical transistor. For $\alpha \approx \pi/2$ one notes the region of negative resistance.

2WM with crossed polarization and 4WM with parallel polarization. Our analytical results are consistent with their numerical findings, except for the case of symmetry breaking noted in the 4WM TG formulas. An example of similarity between the 4WM TG and RG cases is depicted in Fig. 7, which should be compared with the corresponding results of Gu and Yeh.

An interesting field of potential applications for PR rings is photorefractive circuitry. In analogy with the processor electronics, one can understand a piece of PR crystal as a four-pin or an eight-pin optical element (with two or four beam intensities acting as pins at each side of the crystal).[7] Connecting pins would mean closing optical paths through resonators with optical propagators of the form $|r|\exp(i\psi)$. One can construct different PR circuits by interconnecting the pins of one or more than one optical processor. This representation is useful when one is discussing phase-conjugate mirrors with more than one interaction region.[8] Here we display the operation of an optical transistor. The idea is that a weak pump beam can be made to control a much stronger intracavity beam through the PR interaction in the crystal. This is displayed by use of either a 2WM or a 4WM ring in the RG; however (because of symmetries), one can also use TG. The controlling (input) beams are A_{10} and A_{2d}, and the output beams are A_{30} and A_{4d}. If the ring is assumed to operate in the usual common-emitter transistor mode, then the role of the collector current is reserved for I_{4d}, the role for the base current is reserved for I_{2d}, and the roles of the controlling voltages V_{CE} (collector–emitter) and V_{BE} (base–emitter) are reserved for I_{10} and I_{30}. The current gain β (the ratio of I_{4d}/I_{2d}) is obtained from Eqs. (24) and (25):

$$\beta = \frac{\kappa^2}{|r|^2} = \frac{1 - \dfrac{\tau^2}{|r|^2}}{1 - |r|^2}\,. \tag{3}$$

Hence it is connected with the amplification parameter κ. Likewise for the 2WM ring, the role of the currents is played by $|A_{pd}|^2$ and $|B_{s0}|^2$ and the role of the voltages is played by $|B_{p0}|$ and $|A_{sd}|^2$. The volt–ampere characteristics of the device are presented in Fig. 8. It is seen that a remarkable analogy to the operation of an electronic device exists. This analogy can be carried over to the dynamical domain as well.[8] PR rings can mimic the operation of other electronic devices, such as the ones that display negative resistance (thyristors). Because PR crystals allow for parallel processing, there is the possibility of building integrated all-optical PR circuits to be used in optical computers and communications. However, it remains to be seen whether such circuits can be built and usefully implemented in a very demanding field.

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