Unified method for solution of wave equations in photorefractive media

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A unified but simple method for solution of four-wave mixing equations in photorefractive crystals in both transmission and reflection geometries is presented. The method is applied to the problems of double phase conjugation and two-wave mixing with crossed polarizations in cubic crystals.

Wave equations describing the mixing of laser beams in photorefractive crystals thus far have been solved in a number of ways.[1–4] However, the solution procedures for different geometries of the mixing process differ widely. Even the procedures for the same geometry (transmission or reflection) are unrelated. Further, the results all turn out to be rather complicated, despite the simple form of the beginning equations. Finally, the apparent symmetries of the equations have not been used to facilitate the solution process.

We try to improve on this situation by offering a simple universal method that can be used for both geometries and that uses symmetries, yields simple formulas, and provides an explicit procedure for fitting boundary conditions. The method represents a synthesis of previous efforts.[1,3] We apply it to double phase conjugation (DPC) and to the problem of two-wave mixing with crossed polarizations in cubic photorefractive crystals, a problem that has not been solved before to our knowledge.

The geometry of the process is presented in Fig. 1. The mixing of waves in the crystal produces two predominant kinds of diffraction gratings: large-spaced transmission and small-spaced reflection gratings. The grating amplitude for the first kind is given by $Q_T = A_1\overline{A}_4 + \overline{A}_2A_3$, while for the second kind it is $Q_R = A_1\overline{A}_3 + \overline{A}_2A_4$, where the overbar stands for complex conjugation. Slowly varying envelope wave equations are of the form

$$IA_1' = \Gamma Q_T A_4, \qquad IA_4' = -\Gamma \overline{Q}_T A_1, \tag{1}$$

$$IA_2' = \Gamma \overline{Q}_T A_3, \qquad IA_3' = -\Gamma Q_T A_2 \tag{2}$$

for the transmission geometry (TG) and

$$IA_1' = \Gamma Q_R A_3, \qquad IA_3' = \Gamma \overline{Q}_R A_1, \tag{3}$$

$$IA_2' = \Gamma \overline{Q}_R A_4, \qquad IA_4' = \Gamma Q_R A_2 \tag{4}$$

for the reflection geometry (RG). Here $I = \sum |A_1|^2$ is the total intensity, Γ is the coupling constant (real in photorefractives), and the prime is the derivative along the propagation (z) direction. A steady-state, degenerate, plane-wave situation is assumed. The object of the analysis is to solve these equations as a boundary-value problem.

A solution procedure usually consists of two parts: the first part is the method for solving the equations, the second the procedure for fitting boundary conditions. Often the second part is more complicated than the first part. The key to our method is to note, first, that the phase ϕ of the grating amplitude is constant and, second, that because of the symmetries of the equations the fields can be naturally paired and the equations represented in a matrix form. With this in mind, one introduces a convenient new independent variable,

$$\theta' = \frac{\Gamma|Q_{T/R}|}{I} \tag{5}$$

(one θ for the TG and one for the RG) and rewrites Eqs. (1)–(4) by using matrices:

$$\begin{bmatrix} A_1 \\ A_4 \end{bmatrix}' = \mathcal{T}(\phi)\begin{bmatrix} A_1 \\ A_4 \end{bmatrix},$$

$$\begin{bmatrix} A_3 \\ -A_2 \end{bmatrix}' = \mathcal{T}(\phi)\begin{bmatrix} A_3 \\ -A_2 \end{bmatrix}, \tag{6}$$

for the TG, and

$$\begin{bmatrix} A_1 \\ A_3 \end{bmatrix}' = \mathcal{R}(\phi)\begin{bmatrix} A_1 \\ A_3 \end{bmatrix},$$

$$\begin{bmatrix} A_4 \\ A_2 \end{bmatrix}' = \mathcal{R}(\phi)\begin{bmatrix} A_4 \\ A_2 \end{bmatrix}, \tag{7}$$

for the RG. The prime now denotes the derivative with respect to θ. Matrices $\mathcal{T}$ and $\mathcal{R}$ are of the form

$$\mathcal{T}(\phi) = \begin{bmatrix} 0 & \exp(i\phi) \\ -\exp(-i\phi) & 0 \end{bmatrix},$$

$$\mathcal{R}(\phi) = \begin{bmatrix} 0 & \exp(i\phi) \\ \exp(-i\phi) & 0 \end{bmatrix}. \tag{8}$$

In this manner the original nonlinear problem is trans-

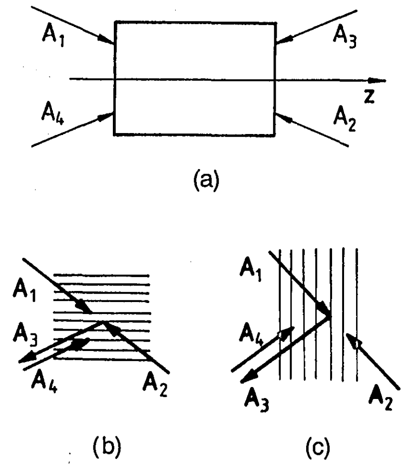

Fig. 1. (a) Geometry of four-wave mixing: (b) TG, (c) RG.

formed into a linear problem, which is easily solved:

$$\begin{bmatrix} A_1 \\ A_4 \end{bmatrix} = \mathcal{M}_T(\phi, \theta - \theta_0)\begin{bmatrix} C_1 \\ C_4 \end{bmatrix},$$
$$\begin{bmatrix} A_3 \\ -A_2 \end{bmatrix} = \mathcal{M}_T(\phi, \theta - \theta_d)\begin{bmatrix} C_3 \\ -C_2 \end{bmatrix}, \tag{9}$$

$$\begin{bmatrix} A_1 \\ A_3 \end{bmatrix} = \mathcal{M}_R(\phi, \theta - \theta_d)\begin{bmatrix} A_{1d} \\ C_3 \end{bmatrix},$$
$$\begin{bmatrix} A_4 \\ A_2 \end{bmatrix} = \mathcal{M}_R(\phi, \theta - \theta_0)\begin{bmatrix} C_4 \\ A_{20} \end{bmatrix}, \tag{10}$$

where C_i are the given boundary values of the four fields (specified at the $z = 0$ and $z = d$ faces of the crystal) and the matrices $\mathcal{M}_T$, $\mathcal{M}_R$ are given by

$$\mathcal{M}_T(\phi, \theta) = \begin{bmatrix} \cos(\theta) & \exp(i\phi)\sin(\theta) \\ -\exp(-i\phi)\sin(\theta) & \cos(\theta) \end{bmatrix}, \tag{11}$$

$$\mathcal{M}_R(\phi, \theta) = \begin{bmatrix} \cosh(\theta) & \exp(i\phi)\sinh(\theta) \\ \exp(-i\phi)\sinh(\theta) & \cosh(\theta) \end{bmatrix}. \tag{12}$$

Thus the original problem is solved if one determines θ as a function of z and θ_0 as a function of boundary values. We note in passing that this representation reveals an SU symmetry of the original problem[5,6] [SU(2) for the TG and SU(1, 1) for the RG]. This symmetry is reflected in the form of the solutions and in the solution procedure. However, the symmetry will be broken along the way. In fact, the first methodological difference between the TG and the RG is already apparent. The pairs of fields in the TG are naturally connected with the boundary conditions, while in the RG they are not. Unspecified (missing) boundary values $A_{20} = A_2(z = 0)$ and $A_{1d} = A_1(z = d)$ are mixed in, and the procedures for fitting boundary conditions will not be symmetric.

A second methodological difference is connected with the solution of Eq. (5). It is easily verified that the total intensity I is constant in the TG case:

$$I = T_1 + T_2, \tag{13}$$

where $T_1 = I_1 + I_4$ and $T_2 = I_2 + I_3$ are the partial intensities. In the RG the total intensity is not conserved, but the total power flow is conserved:

$$F = R_1 + R_2, \tag{14}$$

where $R_1 = I_1 - I_3$, $R_2 = I_2 - I_4$ are the partial flows. Thus the solution of Eq. (5) will be different in the two geometries. This solution is facilitated by the existence of two conserved quantities of higher order:

$$T_3 = F^2 + 4|Q_T|^2, \qquad R_3 = I^2 - 4|Q_R|^2. \tag{15}$$

With the help of these quantities the following expressions are obtained for the magnitudes of the grating amplitude:

$$|Q_T| = \frac{\sqrt{T_3}}{2}\sin(2\theta), \qquad |Q_R| = \frac{\sqrt{R_3}}{2}\sinh(2\theta), \tag{16}$$

and for the flow and the intensity,

$$F = -\sqrt{T_3}\cos(2\theta), \qquad I = \sqrt{R_3}\cosh(2\theta). \tag{17}$$

Now Eq. (5) is integrated:

$$\tan(\theta) = \tan(\theta_0)\exp[(\sqrt{T_3}/I)\Gamma z],$$
$$\sinh(2\theta) = \sinh(2\theta)\exp(\Gamma z), \tag{18}$$

and the solution part of the problem is finished. There remains the boundary-value fitting part.

In the TG, to start with, one finds the expressions for $|Q_d| + |Q_0|$ and $|Q_d| - |Q_0|$:

$$q\cos(\theta_d - \theta_0) + v\sin(\theta_d - \theta_0) = \sqrt{T_3}\sin(\theta_d + \theta_0), \tag{19}$$
$$p\sin(\theta_d - \theta_0) + w\cos(\theta_d - \theta_0) = \sqrt{T_3}\cos(\theta_d + \theta_0), \tag{20}$$

where $q = (\overline{C}_2 C_3 + C_1\overline{C}_4)\exp(-i\phi) +$ c.c., $p = (\overline{C}_2 C_3 - C_1 C_4)\exp(-i\phi) +$ c.c., $v = |C_4|^2 - |C_3|^2 + |C_2|^2 - |C_1|^2$, and $w = |C_4|^2 + |C_3|^2 - |C_2|^2 - |C_1|^2$. Together with the relation

$$\tan(\theta_0) = \alpha\tan(\theta_d), \tag{21}$$

where $\alpha = \exp(-\sqrt{T_3}\,\Gamma d/I)$, a system of three algebraic equations (for θ_d, θ_0, and T_3) is formed. This system is solved as follows.

With the shorthand notation $x = \tan(\theta_d - \theta_0)$, and $y = \tan(\theta_d + \theta_0)$, and with Eq. (21), two quadratic equations for $t = \tan(\theta_0)$ are obtained:

$$t^2 - 2\xi t + \alpha = 0, \qquad t^2 + 2\eta t - \alpha = 0, \tag{22}$$

where $\xi = (1 - \alpha)/2x$ and $\eta = (1 + \alpha)/2y$. Thus

$$t = \xi - \eta = \frac{\alpha}{\xi + \eta}, \tag{23}$$

and the consistency requirement leads to

$$\xi^2 - \eta^2 = \alpha, \tag{24}$$

which is an implicit equation for T_3. This is most easily seen if one squares and adds Eqs. (19) and (20). After some algebra a quadratic equation for x is obtained, whose solution depends only on T_3:

$$x = \frac{c}{T_3 - a} \pm \left[\left(\frac{c}{T_3 - a}\right)^2 - \frac{T_3 - b}{T_3 - a}\right]^{1/2}, \tag{25}$$

where $a = p^2 + v^2$, $b = q^2 + w^2$, and $c = pw + qv$. On the other hand, if one divides Eq. (19) by Eq. (20), a bilinear relation connecting x and y is obtained:

$$y = \frac{q + vx}{px + w}, \qquad x = \frac{q - wy}{py - v}. \tag{26}$$

Thus x, y, and α depend only on T_3, and when Eq. (24) is solved (numerically), the value of T_3 is found for a given set of boundary conditions. Then Eq. (23) yields a value for $\tan(\theta_0)$. Even though this general procedure looks a bit complicated, in applications (as will be seen below) it leads to simple expressions.

The fitting procedure for the RG case proceeds similarly. First Eqs. (10) are used to evaluate the missing boundary values from the known ones:

$$\begin{bmatrix} A_{1d} \\ A_{30} \end{bmatrix} = \mathcal{N}_R(\phi, u)\begin{bmatrix} C_1 \\ C_3 \end{bmatrix}, \qquad \begin{bmatrix} A_{4d} \\ A_{20} \end{bmatrix} = \mathcal{N}_R(\phi, u)\begin{bmatrix} C_4 \\ C_2 \end{bmatrix}, \tag{27}$$

where the matrix $\mathcal{N}_R$ is given by

$$\mathcal{N}_R(\phi, u) = \begin{bmatrix} \operatorname{sech}(u) & \exp(i\phi)\tanh(u) \\ -\exp(-i\phi)\tanh(u) & \operatorname{sech}(u) \end{bmatrix} \tag{28}$$

and $u = \theta_d - \theta_0$ is the so-called grating action. Second, from the definition and from Eqs. (16) two expressions are found for the magnitude of the grating amplitude $|Q_R|$ at $z = 0$:

$$|Q_0| = \tanh(u)\frac{|C|^2}{e - 1} = \operatorname{sech}(u)|q| - \tanh(u)(|C_1|^2 + |C_4|^2), \tag{29}$$

where $|C|^2 = \sum |C_i|^2$, $e = \exp(\Gamma d)$, and here $|q| = |C_1\overline{C}_3 + \overline{C}_2 C_4|$. This yields an expression for $\sinh(u)$:

$$\sinh(u) = \frac{|q|(e - 1)}{e(|C_1|^2 + |C_4|^2) + |C_2|^2 + |C_3|^2}. \tag{30}$$

Finally, from Eqs. (18) and (30), an expression for $\tanh(2\theta_0)$ is obtained:

$$\tanh(2\theta_0) = \frac{\sinh(2u)}{e - \cosh(2u)}. \tag{31}$$

A few remarks are in order. From the expressions derived here it is easy to write expressions of experimental interest, for example, the reflectivity $\rho = A_{30}/\overline{C}_4$:

$$\rho_T = \cos(u)\frac{C_3}{\overline{C}_4} + \exp(i\phi)\sin(u)\frac{C_2}{\overline{C}_4}, \tag{32}$$

$$\rho_R = \operatorname{sech}(u)\frac{C_3}{\overline{C}_4} - \exp(-i\phi)\tanh(u)\frac{C_1}{\overline{C}_4}. \tag{33}$$

Note that the phase of the grating ϕ is also fixed by the boundary conditions:

$$\phi_T = \arg(C_1\overline{C}_4 + \overline{C}_2 C_3), \qquad \phi_R = \arg(C_1\overline{C}_3 + \overline{C}_2 C_4). \tag{34}$$

However, one can always pick one phase arbitrarily, so that ϕ can be set to zero.

The method can be applied to other wave-mixing processes, for example, to DPC. The geometry of DPC is transmissionlike, with special boundary conditions: $C_1 = 0, C_3 = 0$. The process is driven by two incoherent input beams C_2 and C_4, and two conjugated beams A_1 and A_3 are generated in the crystal—hence the name. DPC cannot be realized with reflectionlike gratings. Using Eqs. (9), one immediately writes the solution of DPC:

$$A_1 = C_4 \sin(\theta - \theta_0), \qquad A_3 = C_2 \sin(\theta_d - \theta), \tag{35}$$

$$A_4 = C_4 \cos(\theta - \theta_0), \qquad A_2 = C_2 \cos(\theta_d - \theta), \tag{36}$$

where, according to Eqs. (18),

$$\tan(\theta) = \tan(\theta_0)\exp(a\Gamma z). \tag{37}$$

The parameter $a = \sqrt{T_3}/I = (1 - \alpha)/(1 + \alpha)$ (not to be confused with the a defined previously), which is needed in the specification of both θ and θ_0, is evaluated from Eq. (24):

$$\alpha = \exp(-a\Gamma d) = \frac{1 - a}{1 + a}. \tag{38}$$

This expression is equivalent to the familiar result[1] $a = \tanh(a\Gamma d/2)$. The angles θ_0 and θ_d are determined from Eqs. (23) and (21):

$$\tan(\theta_0) = \exp\left(-\frac{a\Gamma d}{2}\right)\left(\frac{a - q^*}{a + q^*}\right)^{1/2}, \qquad \tan(\theta_d) = \exp\left(\frac{a\Gamma d}{2}\right)\left(\frac{a - q^*}{a + q^*}\right)^{1/2}, \tag{39}$$

where $q^* = (|C_4|^2 - |C_2|^2)/(|C_4|^2 + |C_2|^2)$ is the reduced input-beam ratio. The reflectivities on both faces of the crystal are given by

$$\rho_0 = \frac{C_2}{\overline{C}_4}\sin(u), \qquad \rho_d = \frac{C_4}{\overline{C}_2}\sin(u), \tag{40}$$

and everything is known.

As a second example, we apply the method to the problem of two-wave mixing with crossed polarizations. Yeh[7] considered an interesting fast mixing process in cubic crystals with point symmetry $\overline{4}3m$ (such as GaAs). Equations describing this process are of the form

$$IA_s' = -\Gamma Q_T B_p, \qquad IB_p' = \Gamma \overline{Q}_T A_s, \tag{41}$$

$$IB_s' = \Gamma \overline{Q}_T A_p, \qquad IA_p' = -\Gamma Q_T B_s \tag{42}$$

for the TG and

$$IA_s' = \Gamma Q_R B_p, \qquad IB_p' = \Gamma \overline{Q}_R A_s, \tag{43}$$

$$IB_s' = \Gamma \overline{Q}_R A_p, \qquad IA_p' = \Gamma Q_R B_s \tag{44}$$

for the RG. A_s, A_p and B_s, B_p are the orthogonally polarized components of the two beams incident upon the crystal, and $Q_{T/R} = A_s\overline{B}_s + A_p\overline{B}_p$ is the grating amplitude. Yeh succeeded in solving the TG case, using a four-wave mixing method developed by Cronin-Golomb *et al.*[1]; however, the RG case could not be solved. Similar equations were derived by Fisher *et al.*[8] They also succeeded in solving only the TG case. We show how to obtain a complete analytical solution by the method outlined above.

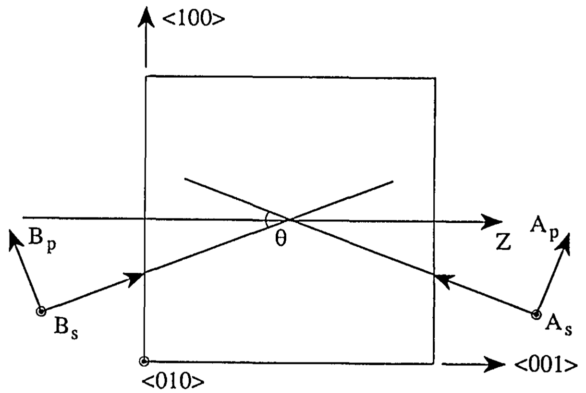

Fig. 2. Geometry of two-wave mixing with crossed polarizations in RG. The angle θ (not to be confused with θ in the text) is assumed to be small.

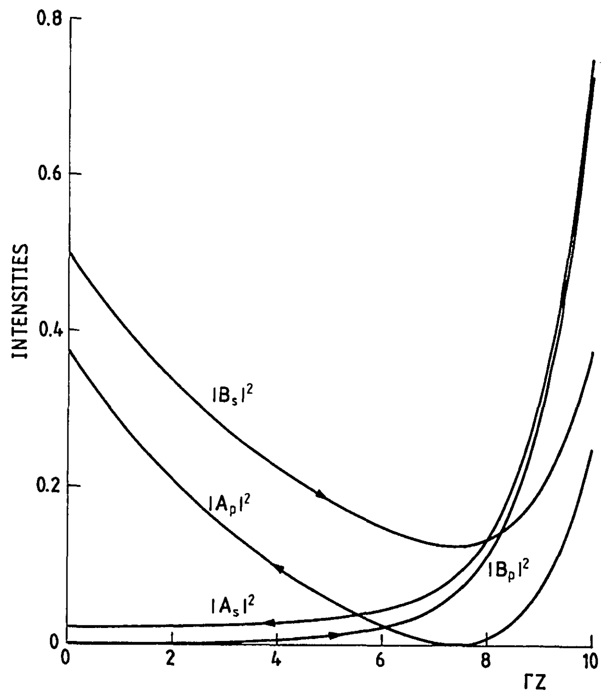

Fig. 3. Two-wave mixing with crossed polarizations in RG. A case with strong coupling and total depletion (and recovery) of one of the beams is depicted. The parameters are $u = 2.455446$, $\theta_d = 1.444779$, $\theta_0 = -1.010667$, and the boundary conditions are $|A_{sd}|^2 = 0.75$, $|A_{pd}|^2 = 0.25$, $|B_{s0}|^2 = 0.5$, $|B_{p0}|^2 = 0$ (arbitrary units).

We are concerned only with the nontrivial case of RG. The geometry of the process is presented in Fig. 2. Equations (43) and (44) are written in the form given by Eqs. (7) if the following identification of the fields is made: $A_s \leftrightarrow A_1$, $A_p \leftrightarrow A_4$, $B_s \leftrightarrow A_2$, $B_p \leftrightarrow A_3$. The solution is then given by the matrix $\mathcal{M}_R$ from Eqs. (10). However, the determination of the angles θ and θ_0 proceeds along different lines, owing to the different form of Q and the different boundary conditions.

The conditions are that the B fields strike the $z = 0$ face of the crystal and the A fields the $z = d$ face. The grating amplitude Q is found to be

$$|Q| = |Q_0|\cosh(2\Theta) + P \sinh(2\Theta), \tag{45}$$

where $\Theta = \theta - \theta_0$ and $|Q_0|$ and P depend only on the grating action $u = \theta_d - \theta_0$. Similarly, the total intensity I is given by

$$I = I_0(u)\cosh(2\Theta) + J(u)\sinh(2\Theta). \tag{46}$$

The determination of $|Q_0|$, P, I_0, and J as functions of u is a simple algebraic problem. Now the equivalent of Eq. (5) is solved:

$$\Gamma z(\Theta) = \frac{I_0|Q_0| - JP}{|Q_0|^2 - P^2}\,\Theta - \frac{I_0 P - J|Q_0|}{2(|Q_0|^2 - P^2)} \times \ln\left[\frac{\mathrm{sch}(2\Theta + \zeta)}{\mathrm{sch}(\zeta)}\right], \tag{47}$$

where the function sch(x) is defined as

$$\mathrm{sch}(x) = \begin{cases} \sinh(x) & |Q_0| < P \\ \cosh(x) & |Q_0| > P \end{cases} \tag{48}$$

and $\tanh(\zeta) = \min(|Q_0|, p)/\max(|Q_0|, P)$.

The fitting of the boundary conditions in this case is simple: One need only determine u. This is done by evaluating Eq. (47) at $z = d$. The value of θ_0 is found from the analog of relations (16) and (17):

$$|R - R_0| + \frac{1}{2}[r\ \mathrm{sech}(u) - I_{BB}\tanh(u)] = \frac{\sqrt{R_3}}{2}\sinh(2\theta), \tag{49}$$

$$I = \sqrt{R_3}\cosh(2\theta), \tag{50}$$

where $R = A_s\overline{B}_p + A_p\overline{B}_s$, $r = (A_{sd}\overline{B}_{p0} + A_{pd}\overline{B}_{s0})\exp(-i\phi) +$ c.c., $\phi = \arg(A_{sd}\overline{B}_{s0} + A_{pd}\overline{B}_{p0})$, and $I_{BB} = 2I_{B0}$. At $z = 0$, relations (49) and (50) yield

$$\tanh(2\theta_0) = \frac{r - I_{BB}\sinh(u)}{I_{AB}\,\mathrm{sech}\,(u) - r\tanh(u)}, \tag{51}$$

where $I_{AB} = I_{Ad} + \cosh(2u)I_{B0}$. This completes the solution. We have compared our analytical solution with the numerical solution[9] and have found perfect agreement. An example of the cross-polarization two-wave mixing process is depicted in Fig. 3.

In summary, we have presented a unified method for solution of wave-mixing equations in photorefractive media. We have shown how the method works by solving in parallel the four-wave mixing problem in both the TG and the RG. We have then applied the method to the problems of DPC and to two-wave mixing with crossed polarizations in RG, which until now have not been solved.

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