Optical soliton perturbation with full nonlinearity in polarization preserving fibers using trial equation method

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This paper obtains optical soliton solution to perturbed nonlinear Schrödinger's equation by trial equation approach. There are nine types of nonlinear fibers studied in this paper. They are Kerr-law, power-law, quadratic-cubic law, parabolic-law, dual-power law, log-law, anti-cubic law, cubic-quintic-septic law and triple-power law. Bright, dark and singular soliton solutions are derived. Additional solutions such as singular periodic solutions also fall out of the integration scheme.

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1. Introduction

Theory of solitons in optical fibers is a very rich area of research in the field of nonlinear optics. Optical soliton molecules are pulses that act as information carriers through optical fibers for trans-continental and trans-oceanic distances. We observe profound progress in the field of nonlinear optics [1-25]. For example, solitons in photonic crystal fibers, diffraction Bragg gratings, dispersion-managed solitons and quasi-linear pulses are a few of the latest advances. There exist still ongoing research activities in optical bullets, spatial solitons and spatio-temporal solitons.

In this work, the nonlinear Schrödinger's equation (NLSE) which includes perturbation terms of certain types is examined. We investigate nine forms of nonlinear media and they are Kerr-law, power-law, quadratic-cubic law, parabolic-law, dual-power law, log-law, anti-cubic law, cubic-quintic-septic law and triple-power law nonlinearity. In order to integrate the perturbed NLSE for each type of nonlinearity, the trial equation method is employed.

The dimensionless form of the nonlinear Schrödinger equation (NLSE) is given by ([12])

$$iq_t + aq_{xx} + bF\left(|q|^2\right)q = 0 \tag{1}$$

where x represents the non-dimensional distance along the fiber while, t represents time in dimensionless form and a and b are real valued constants. The dependent variable $q(x,t)$ is a complex valued function that represents the wave profile. The first term in this equation is temporal evolution, while the coefficient of a is the group velocity dispersion (GVD). The coefficient of b the source of nonlinearity. Solitos are the outcome of a delicate balance between GVD and nonlinearity. The function $F\left(|q|^2\right)q$ is a real-valued algebraic function and is k times continuously differentiable, so that

$$F\left(|q|^2\right)q \in \bigcup_{m,n=1}^{\infty} C^k((-n,n)\times(-m,m); R^2).$$

In presence of perturbation terms, NLSE is modified to ([2],[7],[18],[19])

$$\begin{aligned} &iq_t + aq_{xx} + bF\left(|q|^2\right)q \\ &= i\left[\alpha q_x + \lambda\left(|q|^{2m}q\right)_x + \nu\left(|q|^{2m}\right)_x q\right] \end{aligned} \tag{2}$$

where α is the inter-modal dispersion, λ represents the coefficient of self-steepening for short pulses and ν is the higher-order dispersion coefficient. The parameter m is the full nonlinearity parameter.

2. Succinct overview of trial equation method

In this section we outline the main steps of the trial equation method as following ([1], [4]):

Step-1: Suppose a nonlinear PDE with time-dependent coefficients

$$P(u,u_t,u_x,u_{tt},u_{xt},u_{xx},...)=0 \tag{3}$$

can be converted to an ordinary differential equation (ODE)

$$Q(U,U',U'',U''',...)=0 \tag{4}$$

using a travelling wave hypothesis $u(x,t) = U(\xi)$, $\xi = x - vt$, where $U = U(\xi)$ is an unknown function, Q is a polynomial in the variable U and its derivatives. If all terms contain derivatives, then Eq.(4) is integrated where integration constants are considered zeros.

Step-2: Take the trial equation

$$\left(U'\right)^2 = F(U) = \sum_{l=0}^{N} a_l U^l \tag{5}$$

where a_l, ($l = 0,1,...,N$) are constants to be determined. Substituting Eq.(5) and other derivative terms such as U'' or U''' and so on into Eq.(4) yields a polynomial $G(U)$ of U. According to the balance principle we can determine the value of N. Setting the coefficients of $G(U)$ to zero, we get a system of algebraic equations. Solving this system, we can determine v and values of a_0, a_1, ..., a_N.

Step-3: Rewrite Eq.(5) by the integral form

$$\pm(\xi-\xi_0)=\int\frac{dU}{\sqrt{F(U)}} \tag{6}$$

According to the complete discrimination system of the polynomial, we classify the roots of $F(U)$, and solve the integral Eq.(6). Thus we obtain the exact solutions to Eq.(3).

3. Soliton solutions

In order to solve Eq.(2) by the trial equation method, we use the following wave transformation

$$q(x,t)=U(\xi)e^{i\varphi(x,t)} \tag{7}$$

where $U(\xi)$ represents the shape of the pulse, $\xi = x - vt$ and $\varphi = -\kappa x + \omega t + \theta$. The function $\varphi(x,t)$ is the phase component of the soliton, κ is the soliton frequency, while ω is the wave number, θ is the phase constant and v is the velocity of the soliton.

Substituting Eq.(7) into Eq.(2) and then decomposing into real and imaginary parts yields a pair of relations ([2]). The imaginary part gives

$$v+2a\kappa+\alpha+\left\{(2m+1)\lambda+2m\nu\right\}U^{2m}=0 \tag{8}$$

while the real part gives

$$\begin{aligned}&aU''-\left(\omega+\alpha\kappa+a\kappa^2\right)U-\\&\lambda\kappa U^{2m+1}+bF(U^2)U=0\end{aligned} \tag{9}$$

The imaginary part equation implies

$$v=-2a\kappa-\alpha \tag{10}$$

and

$$(2m+1)\lambda+2m\nu=0 \tag{11}$$

Eq.(10) gives the velocity of the soliton and Eq.(11) gives the constraint relation between the two perturbation terms, while Eq.(9) can be integrated to determine the soliton profile. This form for the velocity remains the same for all types of nonlinearity, $F(s)$.

3.1. Kerr - law

The Kerr-law of nonlinearity originates from the fact that a light wave in an optical fiber faces nonlinear responses from nonharmonic motion of electrons bound in molecules, caused by an external electric field. As a result the induced polarization is not linear in the electric field, but involves higher order terms in electric field amplitude. Even though the nonlinear responses are extremely weak, their effects appear in various ways over long distance propagation that is measured in terms of light wavelength ([2]).

For Kerr-law nonlinearity

$$F(s)=s \tag{12}$$

so that Eq.(2) reduces to

$$\begin{aligned}&iq_t+aq_{xx}+b|q|^2q=\\&i\left[\alpha q_x+\lambda\left(|q|^{2m}q\right)_x+\nu\left(|q|^{2m}\right)_x q\right]\end{aligned} \tag{13}$$

and Eq.(9) simplifies to

$$aU'' - \left(\omega + \alpha\kappa + a\kappa^2\right)U - \lambda\kappa U^{2m+1} + bU^3 = 0 \quad (14)$$

Balancing U'' with U^{2m+1} in Eq.(14), then we get $N = 2m+2$. Using the solution procedure of the trial equation method, we obtain the system of algebraic equations as follows:

U^{2m+1} Coeff.:

$$a(m+1)a_{2m+2} - \lambda\kappa = 0,$$

U^3 Coeff.:

$$2aa_4 + b = 0,$$

U^2 Coeff.:

$$aa_3 = 0,$$

U^1 Coeff.:

$$aa_2 - \omega - \alpha\kappa - a\kappa^2 = 0,$$

U^0 Coeff.:

$$aa_1 = 0.$$

Solving the above system leads to

$$a_1 = 0,\ a_2 = \frac{\omega + \alpha\kappa + a\kappa^2}{a},\ a_3 = 0,$$
$$a_4 = -\frac{b}{2a},\ a_{2m+2} = \frac{\lambda\kappa}{a(m+1)}$$

and also apparent coefficients from Eq.(14)

$$a_5 = a_6 = \ldots = a_{2m+1} = 0$$

where a_0, ω, α, κ, a, b, λ are arbitrary constants.

Substituting these results into Eqs.(5) and (6), we get

$$\pm(\xi - \xi_0) = \int \frac{dU}{\sqrt{a_0 + \left(\frac{\omega + \alpha\kappa + a\kappa^2}{a}\right)U^2 - \frac{b}{2a}U^4 + \frac{\lambda\kappa}{a(m+1)}U^{2m+2}}} \quad (15)$$

In order to carry out the integration of Eq.(15) it is necessary to choose $m = 1$. Thus, with $m = 1$, new Eq.(15) is following:

$$\pm(\xi - \xi_0) = \int \frac{dU}{\sqrt{a_0 + \left(\frac{\omega + \alpha\kappa + a\kappa^2}{a}\right)U^2 + \left(\frac{\lambda\kappa - b}{2a}\right)U^4}} \quad (16)$$

Case 1

If we set $a_0 = 0$ in Eq.(16) and integrating with respect to U, we get the following exact solution of Eq.(13):

$$q(x,t) = \pm\sqrt{\frac{2\left(\omega + \alpha\kappa + a\kappa^2\right)}{b - \lambda\kappa}}\ sech\left[\sqrt{\frac{\omega + \alpha\kappa + a\kappa^2}{a}}\left(x - vt\right)\right]e^{i(-\kappa x + \omega t + \theta)}, \quad (17)$$

$$q(x,t) = \pm\sqrt{\frac{2\left(\omega + \alpha\kappa + a\kappa^2\right)}{\lambda\kappa - b}}\ csch\left[\sqrt{\frac{\omega + \alpha\kappa + a\kappa^2}{a}}\left(x - vt\right)\right]e^{i(-\kappa x + \omega t + \theta)}, \quad (18)$$

where Eq.(17) and Eq.(18) represent bright and singular soliton solutions respectively. These solitons are valid for

$$a\left(\omega + \alpha\kappa + a\kappa^2\right) > 0.$$

$$q(x,t) = \pm\sqrt{\frac{2\left(\omega + \alpha\kappa + a\kappa^2\right)}{b - \lambda\kappa}}\ \sec\left[\sqrt{-\frac{\omega + \alpha\kappa + a\kappa^2}{a}}\left(x - vt\right)\right]e^{i(-\kappa x + \omega t + \theta)}, \quad (19)$$

$$q(x,t) = \pm\sqrt{\frac{2\left(\omega + \alpha\kappa + a\kappa^2\right)}{b - \lambda\kappa}}\ \csc\left[\sqrt{-\frac{\omega + \alpha\kappa + a\kappa^2}{a}}\left(x - vt\right)\right]e^{i(-\kappa x + \omega t + \theta)}, \quad (20)$$

where Eq.(19) and Eq.(20) represent singular periodic solutions. These solutions are valid for

$$a\left(\omega + \alpha\kappa + a\kappa^2\right) < 0.$$

Case 2

If we set $a_0 = \frac{\left(\omega + \alpha\kappa + a\kappa^2\right)^2}{2a\left(\lambda\kappa - b\right)}$ in Eq.(16) and integrating with resect to U, we get the following exact solution of Eq.(13):

$$q(x,t)=\pm\sqrt{\frac{\omega+\alpha\kappa+a\kappa^2}{\lambda\kappa-b}}\tan\left[\sqrt{\frac{\omega+\alpha\kappa+a\kappa^2}{2a}}(x-vt)\right]e^{i(-\kappa x+\omega t+\theta)}, \tag{21}$$

$$q(x,t)=\pm\sqrt{\frac{\omega+\alpha\kappa+a\kappa^2}{\lambda\kappa-b}}\cot\left[\sqrt{\frac{\omega+\alpha\kappa+a\kappa^2}{2a}}(x-vt)\right]e^{i(-\kappa x+\omega t+\theta)}, \tag{22}$$

where Eq.(21) and Eq.(22) represent singular periodic solutions. These solutions are valid for

$$a\left(\omega+\alpha\kappa+a\kappa^2\right)>0.$$

$$q(x,t)=\pm\sqrt{\frac{\omega+\alpha\kappa+a\kappa^2}{b-\lambda\kappa}}\tanh\left[\sqrt{-\frac{\omega+\alpha\kappa+a\kappa^2}{2a}}(x-vt)\right]e^{i(-\kappa x+\omega t+\theta)}, \tag{23}$$

$$q(x,t)=\pm\sqrt{\frac{\omega+\alpha\kappa+a\kappa^2}{b-\lambda\kappa}}\coth\left[\sqrt{-\frac{\omega+\alpha\kappa+a\kappa^2}{2a}}(x-vt)\right]e^{i(-\kappa x+\omega t+\theta)}, \tag{24}$$

where Eq.(23) and Eq.(24) represent dark and singular soliton solutions respectively. These solitons are valid for

$$a\left(\omega+\alpha\kappa+a\kappa^2\right)<0.$$

3.2. Power - law

The power-law nonlinearity is exhibited in various materials including semiconductors. This law also occurs in media for which higher-order photon processes dominate at different intensities. Moreover, in nonlinear plasmas, the power-law solves the problem of small- K condensation in weak turbulence theory. This law is also treated as a generalization to the Kerr-law nonlinearity ([2]).

For power-law nonlinearity

$$F(s)=s^n \tag{25}$$

so that Eq.(2) collapses to

$$iq_t+aq_{xx}+b|q|^{2n}q=i\left[\alpha q_x+\lambda\left(|q|^{2m}q\right)_x+\nu\left(|q|^{2m}\right)_x q\right] \tag{26}$$

The parameter n is in the range $0<n<2$, and in particular $n\neq 2$ since this case leads to a self-focusing singularity. In this case, Eq.(9) simplifies to

$$aU''-\left(\omega+\alpha\kappa+a\kappa^2\right)U-\lambda\kappa U^{2m+1}+bU^{2n+1}=0 \tag{27}$$

To obtain an analytic solution, we use the transformation

$U = V^{\frac{1}{2n}}$ in Eq.(27) to find

$$a\left((1-2n)\left(V'\right)^2+2nVV''\right)-4n^2V^2\left(\omega+\alpha\kappa+a\kappa^2\right)-4\lambda\kappa n^2V^{\frac{m}{n}+2}+4n^2bV^3=0 \tag{28}$$

Balancing VV'' or $\left(V'\right)^2$ with $V^{\frac{m}{n}+2}$ in Eq.(28), then we get $N=\frac{m}{n}+2$. Using the solution procedure of the trial equation method, we obtain the system of algebraic equations as follows:

$V^{\frac{m}{n}+2}$ Coeff.:

$$-4\lambda\kappa n^2+a(1+m)a_{\frac{m}{n}+2}=0,$$

V^3 Coeff.:

$$4n^2b+a(1+n)a_3=0,$$

V^2 Coeff.:

$$-4n^2\left(\omega+\alpha\kappa+a\kappa^2\right)+aa_2=0,$$

V^1 Coeff.:

$$a(1-n)a_1=0,$$

V^0 Coeff.:

$$a(1-2n)a_0=0.$$

Solving the above system leads to

$$a_0=0,\ a_1=0,\ a_2=\frac{4n^2\left(\omega+\alpha\kappa+a\kappa^2\right)}{a},$$

$$a_3=-\frac{4n^2b}{a(1+n)},\ a_{\frac{m}{n}+2}=\frac{4\lambda\kappa n^2}{a(m+1)}$$

and also apparent coefficients from Eq.(28)

$$a_4=a_5=\ldots=a_{\frac{m}{n}+1}=0$$

where ω, α, κ, a, b, λ are arbitrary constants.

Substituting these results into Eqs.(5) and (6), we have

$$\pm(\xi-\xi_0)=\int\frac{dV}{\sqrt{\frac{4n^2\left(\omega+\alpha\kappa+a\kappa^2\right)}{a}V^2-\frac{4n^2b}{a(1+n)}V^3+\frac{4\lambda\kappa n^2}{a(m+1)}V^{\frac{m}{n}+2}}} \quad (29)$$

Case 1

For $m=n$, Eq.(29) reduces to

$$\pm(\xi-\xi_0)=\int\frac{dV}{\sqrt{\frac{4n^2\left(\omega+\alpha\kappa+a\kappa^2\right)}{a}V^2+\frac{4n^2(\lambda\kappa-b)}{a(n+1)}V^3}} \quad (30)$$

Integrating Eq.(30), we obtain the exact solutions of Eq.(26) as follows:

$$q(x,t)=\pm\left\{\sqrt{\frac{(n+1)\left(\omega+\alpha\kappa+a\kappa^2\right)}{b-\lambda\kappa}}\,sech\left[\sqrt{\frac{n^2\left(\omega+\alpha\kappa+a\kappa^2\right)}{a}}(x-vt)\right]\right\}^{\frac{1}{n}} \quad (31)$$

$e^{i(-\kappa x+\omega t+\theta)}$,

$$q(x,t)=\pm\left\{\sqrt{\frac{(n+1)\left(\omega+\alpha\kappa+a\kappa^2\right)}{\lambda\kappa-b}}\,csch\left[\sqrt{\frac{n^2\left(\omega+\alpha\kappa+a\kappa^2\right)}{a}}(x-vt)\right]\right\}^{\frac{1}{n}} \quad (32)$$

$e^{i(-\kappa x+\omega t+\theta)}$,

where Eq.(31) and Eq.(32) represent bright and singular soliton solutions respectively. These solitons are valid for

$$a\left(\omega+\alpha\kappa+a\kappa^2\right)>0.$$

$$q(x,t)=\pm\left\{\sqrt{\frac{(n+1)\left(\omega+\alpha\kappa+a\kappa^2\right)}{\lambda\kappa-b}}\,\sec\left[\sqrt{-\frac{n^2\left(\omega+\alpha\kappa+a\kappa^2\right)}{a}}(x-vt)\right]\right\}^{\frac{1}{n}} \quad (33)$$

$e^{i(-\kappa x+\omega t+\theta)}$,

$$q(x,t)=\pm\left\{\sqrt{\frac{(n+1)\left(\omega+\alpha\kappa+a\kappa^2\right)}{\lambda\kappa-b}}\,\csc\left[\sqrt{-\frac{n^2\left(\omega+\alpha\kappa+a\kappa^2\right)}{a}}(x-vt)\right]\right\}^{\frac{1}{n}} \quad (34)$$

$e^{i(-\kappa x+\omega t+\theta)}$,

where Eq.(33) and Eq.(34) represent singular periodic solutions. These solutions are valid for

$$a\left(\omega+\alpha\kappa+a\kappa^2\right)<0.$$

Case 2

For $m=2n$, Eq.(29) reduces to

$$\pm(\xi-\xi_0)=\int\frac{dV}{\sqrt{\frac{4n^2\left(\omega+\alpha\kappa+a\kappa^2\right)}{a}V^2-\frac{4n^2b}{a(1+n)}V^3+\frac{4\lambda\kappa n^2}{a(2n+1)}V^4}} \quad (35)$$

Eq.(35) can be integrated with respect to V if we set

$$\omega+\alpha\kappa+a\kappa^2=\frac{(2n+1)b^2}{4\lambda\kappa(n+1)^2}$$

Thus, we obtain exact solutions of Eq.(26):

$$q(x,t)=\left\{\frac{(2n+1)b}{4\lambda\kappa(n+1)}\left[1\pm\tanh\left(\sqrt{\frac{n^2(2n+1)b^2}{4a\lambda\kappa(n+1)^2}}(x-vt)\right)\right]\right\}^{\frac{1}{2n}} \quad (36)$$

$e^{i(-\kappa x+\omega t+\theta)}$,

$$q(x,t)=\left\{\frac{(2n+1)b}{4\lambda\kappa(n+1)}\left[1\pm\coth\left(\sqrt{\frac{n^2(2n+1)b^2}{4a\lambda\kappa(n+1)^2}}(x-vt)\right)\right]\right\}^{\frac{1}{2n}} \quad (37)$$

$e^{i(-\kappa x+\omega t+\theta)}$,

where Eq.(36) and Eq.(37) represent dark and singular soliton solutions respectively. These solitons are valid for

$$a\lambda\kappa>0.$$

3.3 Quadratic-Cubic Law

This nonlinearity first appeared in 2011 ([3]). The general form can be written as

$$F(s) = b_1\sqrt{s} + b_2 s$$

where b_1 and b_2 are constants. The governing NLSE therefore is:

$$iq_t + aq_{xx} + \left(b_1|q| + b_2|q|^2\right)q = i\left[\alpha q_x + \lambda\left(|q|^{2m}q\right)_x + \nu\left(|q|^{2m}\right)_x q\right] \tag{38}$$

In this case, Eq.(9) simplifies to

$$aU'' - \left(\omega + \alpha\kappa + a\kappa^2\right)U - \lambda\kappa U^{2m+1} + b_1U^2 + b_2U^3 = 0 \tag{39}$$

Balancing U'' with U^{2m+1} in Eq.(39), then we get $N = 2m+2$. Using the solution procedure of the trial equation method, we obtain the system of algebraic equations as follows:

U^{2m+1} Coeff.:

$$a(m+1)a_{2m+2} - \lambda\kappa = 0,$$

U^3 Coeff.:

$$2aa_4 + b_2 = 0,$$

U^2 Coeff.:

$$\frac{3aa_3}{2} + b_1 = 0,$$

U^1 Coeff.:

$$aa_2 - \omega - \alpha\kappa - a\kappa^2 = 0,$$

U^0 Coeff.:

$$aa_1 = 0.$$

Solving the above system leads to

$$a_1 = 0,\ a_2 = \frac{\omega + \alpha\kappa + a\kappa^2}{a},\ a_3 = -\frac{2b_1}{3a},$$
$$a_4 = -\frac{b_2}{2a},\ a_{2m+2} = \frac{\lambda\kappa}{a(m+1)}$$

and also apparent coefficients from Eq.(39)

$$a_5 = a_6 = \ldots = a_{2m+1} = 0$$

where a_0, ω, α, κ, a, b_1, b_2, λ are arbitrary constants.

Substituting these results into Eqs.(5) and (6), we get

$$\pm(\xi - \xi_0) = \int \frac{dU}{\sqrt{a_0 + \left(\frac{\omega + \alpha\kappa + a\kappa^2}{a}\right)U^2 - \frac{2b_1}{3a}U^3 - \frac{b_2}{2a}U^4 + \frac{\lambda\kappa}{a(m+1)}U^{2m+2}}} \tag{40}$$

In order to carry out the integration of Eq.(40) it is necessary to choose $m = 1$. Thus, with $m = 1$, new Eq.(40) is following:

$$\pm(\xi - \xi_0) = \int \frac{dU}{\sqrt{a_0 + \left(\frac{\omega + \alpha\kappa + a\kappa^2}{a}\right)U^2 - \frac{2b_1}{3a}U^3 + \left(\frac{\lambda\kappa - b_2}{2a}\right)U^4}} \tag{41}$$

Case 1

Eq.(41) can be integrated with respect to U if we set

$$a_0 = 0,\ \omega + \alpha\kappa + a\kappa^2 = \frac{2b_1^2}{9(\lambda\kappa - b_2)}$$

Thus, we obtain exact solutions of Eq.(38):

$$q(x,t) = \frac{b_1}{3(\lambda\kappa - b_2)}\left\{1 \pm \tanh\left[\sqrt{\frac{b_1^2}{18a(\lambda\kappa - b_2)}}(x - vt)\right]\right\} e^{i(-\kappa x + \omega t + \theta)}, \tag{42}$$

$$q(x,t) = \frac{b_1}{3(\lambda\kappa - b_2)}\left\{1 \pm \coth\left[\sqrt{\frac{b_1^2}{18a(\lambda\kappa - b_2)}}(x - vt)\right]\right\} e^{i(-\kappa x + \omega t + \theta)}, \tag{43}$$

where Eq.(42) and Eq.(43) represent dark and singular soliton solution respectively. These solitons are valid for

$$a(\lambda\kappa - b_2) > 0.$$

Case 2

Eq.(41) can be integrated with respect to U if we set

$$a_0 = 0,\ b_2 = \lambda\kappa$$

Thus, we obtain exact solutions of Eq.(38):

$$q(x,t)=\pm\left\{\begin{array}{l}\dfrac{3\left(\omega+\alpha\kappa+a\kappa^2\right)}{2b_1}\\ sech^2\left[\sqrt{\dfrac{\omega+\alpha\kappa+a\kappa^2}{4a}}\left(x-vt\right)\right]\end{array}\right\}e^{i(-\kappa x+\omega t+\theta)},\quad(44)$$

$$q(x,t)=\pm\left\{\begin{array}{l}-\dfrac{3\left(\omega+\alpha\kappa+a\kappa^2\right)}{2b_1}\\ csch^2\left[\sqrt{\dfrac{\omega+\alpha\kappa+a\kappa^2}{4a}}\left(x-vt\right)\right]\end{array}\right\}e^{i(-\kappa x+\omega t+\theta)},\quad(45)$$

where Eq.(44) and Eq.(45) represent bright and singular soliton solutions respectively. These solitons are valid for

$$a\left(\omega+\alpha\kappa+a\kappa^2\right)>0.$$

$$q(x,t)=\pm\left\{\begin{array}{l}-\dfrac{3\left(\omega+\alpha\kappa+a\kappa^2\right)}{2b_1}\\ sec^2\left[\sqrt{-\dfrac{\omega+\alpha\kappa+a\kappa^2}{4a}}\left(x-vt\right)\right]\end{array}\right\}e^{i(-\kappa x+\omega t+\theta)},\quad(46)$$

$$q(x,t)=\pm\left\{\begin{array}{l}-\dfrac{3\left(\omega+\alpha\kappa+a\kappa^2\right)}{2b_1}\\ csc^2\left[\sqrt{-\dfrac{\omega+\alpha\kappa+a\kappa^2}{4a}}\left(x-vt\right)\right]\end{array}\right\}\quad(47)$$
$$e^{i(-\kappa x+\omega t+\theta)},$$

where Eq.(46) and Eq.(47) represent singular periodic solutions. These solutions are valid for

$$a\left(\omega+\alpha\kappa+a\kappa^2\right)<0.$$

3.4. Parabolic-Law

The collapse of two-and three-dimensional optical beams in a Kerr-law medium was considered as a means of producing high electric field strengths. It was observed that the inclusion of a saturable nonlinearity could halt the singular collapse, thus causing the formation of an optical beam that propagates without changing its temporal or spatial shape, being held together by nonlinear effects. Therefore, it is of interest to consider nonlinearities higher than the third-order to obtain some knowledge of the diameter of the self-trapped beam. There was little or no attention paid to the propagation of optical beams in the fifth-order nonlinear media, since no analytic solutions existed and it seemed that the chances of finding any material with significant fifth-order effects were low. However, recent new results have reignited interest in this area ([2]).

For parabolic-law nonlinearity,

$$F(s)=b_3s+b_4s^2\quad(48)$$

where b_3 and b_4 are constants. Therefore, Eq.(2) takes the form

$$\begin{array}{l}iq_t+aq_{xx}+\left(b_3\left|q\right|^2+b_4\left|q\right|^4\right)q=\\ i\left[\alpha q_x+\lambda\left(\left|q\right|^{2m}q\right)_x+\nu\left(\left|q\right|^{2m}\right)_xq\right]\end{array}\quad(49)$$

In this case, Eq.(9) simplifies to

$$\begin{array}{l}aU''-\left(\omega+\alpha\kappa+a\kappa^2\right)U-\\ \lambda\kappa U^{2m+1}+b_3U^3+b_4U^5=0\end{array}\quad(50)$$

By using transformation $U=V^{\frac{1}{2}}$, Eq.(50) becomes

$$\begin{array}{l}a\left(-\left(V'\right)^2+2VV''\right)-4V^2\left(\omega+\alpha\kappa+a\kappa^2\right)-\\ 4\lambda\kappa V^{m+2}+4\left(b_3V^3+b_4V^4\right)=0\end{array}\quad(51)$$

Balancing VV'' or $\left(V'\right)^2$ with V^{m+2} in Eq.(51), then we get $N=m+2$. Using the solution procedure of the trial equation method, we obtain the system of algebraic equations as follows:

V^{m+2} Coeff.:

$$-4\lambda\kappa+a(m+1)a_{m+2}=0,$$

V^4 Coeff.:

$$4b_4+3aa_4=0,$$

V^3 Coeff.:

$$4b_3+2aa_3=0,$$

V^2 Coeff.:

$$-4\left(\omega+\alpha\kappa+a\kappa^2\right)+aa_2=0,$$

V^0 Coeff.:

$$aa_0=0.$$

Solving the above system leads to

$$a_0 = 0,\ a_2 = \frac{4\left(\omega+\alpha\kappa+a\kappa^2\right)}{a},\ a_3 = -\frac{2b_3}{a},$$

$$a_4 = -\frac{4b_4}{3a},\ a_{m+2} = \frac{4\lambda\kappa}{a(m+1)}$$

and also apparent coefficients from Eq.(51)

$$a_5 = a_6 = \ldots = a_{m+1} = 0$$

where a_1, ω, α, κ, a, b_3, b_4, λ are arbitrary constants.

Substituting these results into Eqs.(5) and (6), we get

$$\pm\left(\xi-\xi_0\right) = \int \frac{dV}{\sqrt{a_1V + \frac{4\left(\omega+\alpha\kappa+a\kappa^2\right)}{a}V^2 - \frac{2b_3}{a}V^3 - \frac{4b_4}{3a}V^4 + \frac{4\lambda\kappa}{a(m+1)}V^{m+2}}} \tag{52}$$

Case 1

For $m=1$, Eq.(52) reduces to

$$\pm\left(\xi-\xi_0\right) = \int \frac{dV}{\sqrt{a_1V + \frac{4\left(\omega+\alpha\kappa+a\kappa^2\right)}{a}V^2 + \frac{2\left(\lambda\kappa-b_3\right)}{a}V^3 - \frac{4b_4}{3a}V^4}} \tag{53}$$

Case 1.1

Eq.(53) can be integrated with respect to V if we set

$$a_1 = 0,\ \omega+\alpha\kappa+a\kappa^2 = -\frac{3\left(\lambda\kappa-b_3\right)^2}{16b_4}$$

Thus, we obtain exact solutions of Eq.(49):

$$q(x,t) = \sqrt{\frac{3\left(\lambda\kappa-b_3\right)}{8b_4}} \left\{1\pm\tanh\left[\sqrt{-\frac{3\left(\lambda\kappa-b_3\right)^2}{16ab_4}}\left(x-vt\right)\right]\right\}^{\frac{1}{2}} e^{i\left(-\kappa x+\omega t+\theta\right)}, \tag{54}$$

$$q(x,t) = \sqrt{\frac{3\left(\lambda\kappa-b_3\right)}{8b_4}} \left\{1\pm\coth\left[\sqrt{-\frac{3\left(\lambda\kappa-b_3\right)^2}{16ab_4}}\left(x-vt\right)\right]\right\}^{\frac{1}{2}} e^{i\left(-\kappa x+\omega t+\theta\right)}, \tag{55}$$

where Eq.(54) and Eq.(55) represent dark and singular soliton solution respectively. These solitons are valid for

$$ab_4 < 0.$$

Case 1.2

Eq.(53) can be integrated with respect to V if we set

$$a_1 = 0,\ b_3 = \lambda\kappa$$

Thus, we obtain exact solutions of Eq.(49):

$$q(x,t) = \pm\left\{\sqrt{\frac{3\left(\omega+\alpha\kappa+a\kappa^2\right)}{b_4}}\, sech\left[\sqrt{\frac{4\left(\omega+\alpha\kappa+a\kappa^2\right)}{a}}\left(x-vt\right)\right]\right\}^{\frac{1}{2}} e^{i\left(-\kappa x+\omega t+\theta\right)}, \tag{56}$$

$$q(x,t) = \pm\left\{\sqrt{-\frac{3\left(\omega+\alpha\kappa+a\kappa^2\right)}{b_4}}\, csch\left[\sqrt{\frac{4\left(\omega+\alpha\kappa+a\kappa^2\right)}{a}}\left(x-vt\right)\right]\right\}^{\frac{1}{2}} e^{i\left(-\kappa x+\omega t+\theta\right)}, \tag{57}$$

where Eq.(56) and Eq.(57) represent bright and singular soliton solutions respectively. These solitons are valid for

$$a\left(\omega+\alpha\kappa+a\kappa^2\right) > 0.$$

$$q(x,t) = \pm\left\{\sqrt{\frac{3\left(\omega+\alpha\kappa+a\kappa^2\right)}{b_4}}\, sec\left[\sqrt{-\frac{4\left(\omega+\alpha\kappa+a\kappa^2\right)}{a}}\left(x-vt\right)\right]\right\}^{\frac{1}{2}} e^{i\left(-\kappa x+\omega t+\theta\right)}, \tag{58}$$

$$q(x,t)=\pm\left\{\sqrt{\frac{3\left(\omega+\alpha\kappa+a\kappa^2\right)}{b_4}}\ csc\left[\sqrt{-\frac{4\left(\omega+\alpha\kappa+a\kappa^2\right)}{a}}\left(x-vt\right)\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta)}, \quad (59)$$

where Eq.(58) and Eq.(59) represent singular periodic solutions. These solutions are valid for

$$a\left(\omega+\alpha\kappa+a\kappa^2\right)<0.$$

Case 2

For $m=2$, Eq.(52) reduces to

$$\pm\left(\xi-\xi_0\right)=\int\frac{dV}{\sqrt{a_1V+\frac{4\left(\omega+\alpha\kappa+a\kappa^2\right)}{a}V^2-\frac{2b_3}{a}V^3+\frac{4\left(\lambda\kappa-b_4\right)}{3a}V^4}} \quad (60)$$

Case 2.1

Eq.(60) can be integrated with respect to V if we set

$$a_1=0,\ \omega+\alpha\kappa+a\kappa^2=\frac{3b_3^2}{16\left(\lambda\kappa-b_4\right)}$$

Thus, we obtain exact solutions of Eq.(49):

$$q(x,t)=\sqrt{\frac{3b_3}{8\left(\lambda\kappa-b_4\right)}}\left\{1\pm\tanh\left[\sqrt{\frac{3b_3^2}{16a\left(\lambda\kappa-b_4\right)}}\left(x-vt\right)\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta)}, \quad (61)$$

$$q(x,t)=\sqrt{\frac{3b_3}{8\left(\lambda\kappa-b_4\right)}}\left\{1\pm\coth\left[\sqrt{\frac{3b_3^2}{16a\left(\lambda\kappa-b_4\right)}}\left(x-vt\right)\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta)}, \quad (62)$$

where Eq.(61) and Eq.(62) represent dark and singular soliton solution respectively. These solitons are valid for

$$a\left(\lambda\kappa-b_4\right)>0.$$

Case 2.2

Eq.(60) can be integrated with respect to V if we set

$$a_1=0,\ b_4=\lambda\kappa$$

Thus, we obtain exact solutions of Eq.(49):

$$q(x,t)=\pm\sqrt{\frac{2\left(\omega+\alpha\kappa+a\kappa^2\right)}{b_3}}\ sech\left[\sqrt{\frac{\left(\omega+\alpha\kappa+a\kappa^2\right)}{a}}\left(x-vt\right)\right]e^{i(-\kappa x+\omega t+\theta)}, \quad (63$$

$$q(x,t)=\pm\sqrt{-\frac{2\left(\omega+\alpha\kappa+a\kappa^2\right)}{b_3}}\ csch\left[\sqrt{\frac{\left(\omega+\alpha\kappa+a\kappa^2\right)}{a}}\left(x-vt\right)\right]e^{i(-\kappa x+\omega t+\theta)}, \quad (64)$$

where Eq.(63) and Eq.(64) represent bright and singular soliton solutions respectively. These solitons are valid for

$$a\left(\omega+\alpha\kappa+a\kappa^2\right)>0.$$

$$q(x,t)=\pm\sqrt{-\frac{2\left(\omega+\alpha\kappa+a\kappa^2\right)}{b_3}}\ sec\left[\sqrt{-\frac{\left(\omega+\alpha\kappa+a\kappa^2\right)}{a}}\left(x-vt\right)\right]e^{i(-\kappa x+\omega t+\theta)}, \quad (65)$$

$$q(x,t)=\pm\sqrt{-\frac{2\left(\omega+\alpha\kappa+a\kappa^2\right)}{b_3}}\ csc\left[\sqrt{-\frac{\left(\omega+\alpha\kappa+a\kappa^2\right)}{a}}\left(x-vt\right)\right]e^{i(-\kappa x+\omega t+\theta)}, \quad (66)$$

where Eq.(65) and Eq.(66) represent singular periodic solutions. These solutions are valid for

$$a\left(\omega+\alpha\kappa+a\kappa^2\right)<0.$$

3.5. Dual-Power Law

This model is used to describe the saturation of the nonlinear refractive index. Also, this serves as a basic model to describe the solitons in photovoltaic-photorefractive materials such as LiNbO3 ([2]).

For dual-power law nonlinearity,

$$F(s) = b_5 s^n + b_6 s^{2n} \tag{67}$$

where b_5 and b_6 are constants. Therefore, Eq.(2) reduces to

$$\begin{aligned} & iq_t + aq_{xx} + \left(b_5 |q|^{2n} + b_6 |q|^{4n}\right) q = \\ & i\left[\alpha q_x + \lambda\left(|q|^{2m} q\right)_x + \nu\left(|q|^{2m}\right)_x q\right] \end{aligned} \tag{68}$$

In this case, Eq.(9) simplifies to

$$\begin{aligned} & aU'' - \left(\omega + \alpha\kappa + a\kappa^2\right)U - \\ & \lambda\kappa U^{2m+1} + b_5 U^{2n+1} + b_6 U^{4n+1} = 0 \end{aligned} \tag{69}$$

By using transformation $U = V^{\frac{1}{2n}}$, Eq.(69) becomes

$$\begin{aligned} & a\left((1-2n)\left(V'\right)^2 + 2nVV''\right) - \\ & 4n^2 V^2 \left(\omega + \alpha\kappa + a\kappa^2\right) - \\ & 4\lambda\kappa n^2 V^{\frac{m}{n}+2} + 4n^2\left(b_5 V^3 + b_6 V^4\right) = 0 \end{aligned} \tag{70}$$

Balancing VV'' or $\left(V'\right)^2$ with $V^{\frac{m}{n}+2}$ in Eq.(70), then we get $N = \frac{m}{n} + 2$. Using the solution procedure of the trial equation method, we obtain the system of algebraic equations as follows:

$V^{\frac{m}{n}+2}$ Coeff.:

$$-4n^2\lambda\kappa + a(1+m)a_{\frac{m}{n}+2} = 0,$$

V^4 Coeff.:

$$4n^2 b_6 + a(1+2n)a_4 = 0,$$

V^3 Coeff.:

$$4n^2 b_5 + a(1+n)a_3 = 0,$$

V^2 Coeff.:

$$-4n^2\left(\omega + \alpha\kappa + a\kappa^2\right) + aa_2 = 0,$$

V^1 Coeff.:

$$a(1-n)a_1 = 0,$$

V^0 Coeff.:

$$a(1-2n)a_0 = 0.$$

Solving the above system leads to

$$a_0 = 0,\ a_1 = 0,\ a_2 = \frac{4n^2\left(\omega + \alpha\kappa + a\kappa^2\right)}{a},$$

$$a_3 = -\frac{4n^2 b_5}{a(1+n)},\ a_4 = -\frac{4n^2 b_6}{a(1+2n)},\ a_{\frac{m}{n}+2} = \frac{4n^2\lambda\kappa}{a(m+1)}$$

and also apparent coefficients from Eq.(70)

$$a_5 = a_6 = a_7 = \ldots = a_{\frac{m}{n}+1} = 0$$

where ω, α, κ, a, b_5, b_6, λ are arbitrary constants.

Substituting these results into Eqs.(5) and (6), we get

$$\pm\left(\xi - \xi_0\right) = \int \frac{dV}{\sqrt{\frac{4n^2\left(\omega + \alpha\kappa + a\kappa^2\right)}{a}V^2 - \frac{4n^2 b_5}{a(1+n)}V^3 - \frac{4n^2 b_6}{a(1+2n)}V^4 + \frac{4n^2\lambda\kappa}{a(m+1)}V^{\frac{m}{n}+2}}} \tag{71}$$

<u>Case 1</u>

For $m = n$, Eq.(71) reduces to

$$\pm\left(\xi - \xi_0\right) = \int \frac{dV}{\sqrt{\frac{4n^2\left(\omega + \alpha\kappa + a\kappa^2\right)}{a}V^2 + \frac{4n^2\left(\lambda\kappa - b_5\right)}{a(n+1)}V^3 - \frac{4n^2 b_6}{a(1+2n)}V^4}} \tag{72}$$

<u>Case 1.1</u>

Eq.(72) can be integrated with respect to V if we set

$$\omega + \alpha\kappa + a\kappa^2 = -\frac{(\lambda\kappa - b_5)^2(1+2n)}{4(n+1)^2 b_6}$$

Thus, we obtain exact solutions of Eq.(68):

$$q(x,t)=\left\{\frac{(\lambda\kappa-b_5)(1+2n)}{4(n+1)b_6}\left[1\pm\tanh\left(\sqrt{-\frac{n^2(\lambda\kappa-b_5)^2(1+2n)}{4(n+1)^2ab_6}}(x-vt)\right)\right]\right\}^{\frac{1}{2n}} e^{i(-\kappa x+\omega t+\theta)}, \tag{73}$$

$$q(x,t)=\left\{\frac{(\lambda\kappa-b_5)(1+2n)}{4(n+1)b_6}\left[1\pm\coth\left(\sqrt{-\frac{n^2(\lambda\kappa-b_5)^2(1+2n)}{4(n+1)^2ab_6}}(x-vt)\right)\right]\right\}^{\frac{1}{2n}} e^{i(-\kappa x+\omega t+\theta)}, \tag{74}$$

where Eq.(73) and Eq.(74) represent dark and singular soliton solution respectively. These solitons are valid for

$$ab_6<0.$$

Case 1.2

Eq.(72) can be integrated with respect to V if we set

$$b_5=\lambda\kappa$$

Thus, we obtain exact solutions of Eq.(68):

$$q(x,t)=\pm\left\{\sqrt{\frac{(2n+1)(\omega+\alpha\kappa+a\kappa^2)}{b_6}}\,sech\left[\sqrt{\frac{4n^2(\omega+\alpha\kappa+a\kappa^2)}{a}}(x-vt)\right]\right\}^{\frac{1}{2n}} e^{i(-\kappa x+\omega t+\theta)}, \tag{75}$$

$$q(x,t)=\pm\left\{\sqrt{-\frac{(2n+1)(\omega+\alpha\kappa+a\kappa^2)}{b_6}}\,csch\left[\sqrt{\frac{4n^2(\omega+\alpha\kappa+a\kappa^2)}{a}}(x-vt)\right]\right\}^{\frac{1}{2n}} e^{i(-\kappa x+\omega t+\theta)}, \tag{76}$$

where Eq.(75) and Eq.(76) represent bright and singular soliton solutions respectively. These solitons are valid for

$$a(\omega+\alpha\kappa+a\kappa^2)>0.$$

$$q(x,t)=\pm\left\{\sqrt{\frac{(2n+1)(\omega+\alpha\kappa+a\kappa^2)}{b_6}}\,sec\left[\sqrt{-\frac{4n^2(\omega+\alpha\kappa+a\kappa^2)}{a}}(x-vt)\right]\right\}^{\frac{1}{2n}} e^{i(-\kappa x+\omega t+\theta)}, \tag{77}$$

$$q(x,t)=\pm\left\{\sqrt{\frac{(2n+1)(\omega+\alpha\kappa+a\kappa^2)}{b_6}}\,csc\left[\sqrt{-\frac{4n^2(\omega+\alpha\kappa+a\kappa^2)}{a}}(x-vt)\right]\right\}^{\frac{1}{2n}} e^{i(-\kappa x+\omega t+\theta)}, \tag{78}$$

where Eq.(77) and Eq.(78) represent singular periodic solutions. These solutions are valid for

$$a(\omega+\alpha\kappa+a\kappa^2)<0.$$

Case 2

For $m=2n$, Eq.(71) reduces to

$$\pm(\xi-\xi_0)=\int\frac{dV}{\sqrt{\frac{4n^2(\omega+\alpha\kappa+a\kappa^2)}{a}V^2-\frac{4n^2b_5}{a(1+n)}V^3+\frac{4n^2(\lambda\kappa-b_6)}{a(1+2n)}V^4}} \tag{79}$$

Case 2.1

Eq.(79) can be integrated with respect to V if we set

$$\omega+\alpha\kappa+a\kappa^2=\frac{(2n+1)b_5^2}{4(n+1)^2(\lambda\kappa-b_6)}$$

Thus, we obtain exact solutions of Eq.(68):

$$q(x,t)=\left\{\frac{(2n+1)b_5}{4(n+1)(\lambda\kappa-b_6)}\left[1\pm\tanh\left(\sqrt{\frac{n^2(2n+1)b_5^2}{4a(n+1)^2(\lambda\kappa-b_6)}}(x-vt)\right)\right]\right\}^{\frac{1}{2n}} e^{i(-\kappa x+\omega t+\theta)}, \tag{80}$$

$$q(x,t)=\left\{\frac{(2n+1)b_5}{4(n+1)(\lambda\kappa-b_6)}\left[1\pm\coth\left(\sqrt{\frac{n^2(2n+1)b_5^2}{4a(n+1)^2(\lambda\kappa-b_6)}}(x-vt)\right)\right]\right\}^{\frac{1}{2n}} e^{i(-\kappa x+\omega t+\theta)}, \tag{81}$$

where Eq.(80) and Eq.(81) represent dark and singular soliton solution respectively. These solitons are valid for

$$a(\lambda\kappa-b_6)>0.$$

Case 2.2

Eq.(79) can be integrated with respect to V if we set

$$b_6=\lambda\kappa$$

Thus, we obtain exact solutions of Eq.(68):

$$q(x,t)=\pm\left\{\sqrt{\frac{(n+1)(\omega+\alpha\kappa+a\kappa^2)}{b_5}}\, sech\left[\sqrt{\frac{n^2(\omega+\alpha\kappa+a\kappa^2)}{a}}(x-vt)\right]\right\}^{\frac{1}{n}} e^{i(-\kappa x+\omega t+\theta)}, \tag{82}$$

$$q(x,t)=\pm\left\{\sqrt{-\frac{(n+1)(\omega+\alpha\kappa+a\kappa^2)}{b_5}}\, csch\left[\sqrt{\frac{n^2(\omega+\alpha\kappa+a\kappa^2)}{a}}(x-vt)\right]\right\}^{\frac{1}{n}} e^{i(-\kappa x+\omega t+\theta)}, \tag{83}$$

where Eq.(82) and Eq.(83) represent bright and singular soliton solutions respectively. These solitons are valid for

$$a(\omega+\alpha\kappa+a\kappa^2)>0.$$

$$q(x,t)=\pm\left\{\sqrt{-\frac{(n+1)(\omega+\alpha\kappa+a\kappa^2)}{b_5}}\, sec\left[\sqrt{-\frac{n^2(\omega+\alpha\kappa+a\kappa^2)}{a}}(x-vt)\right]\right\}^{\frac{1}{n}} e^{i(-\kappa x+\omega t+\theta)}, \tag{84}$$

$$q(x,t)=\pm\left\{\sqrt{-\frac{(n+1)(\omega+\alpha\kappa+a\kappa^2)}{b_5}}\, csc\left[\sqrt{-\frac{n^2(\omega+\alpha\kappa+a\kappa^2)}{a}}(x-vt)\right]\right\}^{\frac{1}{n}} e^{i(-\kappa x+\omega t+\theta)}, \tag{85}$$

where Eq.(84) and Eq.(85) represent singular periodic solutions. These solutions are valid for

$$a(\omega+\alpha\kappa+a\kappa^2)<0.$$

3.6. Log – Law

This log-law nonlinearity arises in various fields of contemporary physics. It allows closed form exact expressions for stationary Gaussian beams (Gaussons) as well as for periodic and quasiperiodic regimes of the beam evolution. The advantage of this model is that the radiation from the periodic soliton is absent as the linearized problem has a discrete spectrum only ([2]).

For case of log-law nonlinearity,

$$F(s)=\ln s \tag{86}$$

so that Eq.(2) collapses to

$$iq_t+aq_{xx}+b\ln\left(|q|^2\right)q = i\left[\alpha q_x+\lambda\left(|q|^{2m}q\right)_x+\nu\left(|q|^{2m}\right)_x q\right] \tag{87}$$

In this case, Eq.(9) simplifies to

$$aU'' - \left(\omega + \alpha\kappa + a\kappa^2\right)U - \lambda\kappa U^{2m+1} + 2bU \ln U = 0 \quad (88)$$

To obtain an analytic solution, we use the transformation $U = e^V$ in Eq.(88) to find

$$a\left(\left(V'\right)^2 + V''\right) - \omega - \alpha\kappa - a\kappa^2 - \lambda\kappa e^{2mV} + 2bV = 0 \quad (89)$$

In order to carry out the balancing procedure in Eq.(89), it is helpful to set $\lambda = 0$. This indicates that the perturbed NLSE with log-law nonlinearity can be integrated only when the self- steepening term is not present. In this case the perturbed NLSE is given by

$$iq_t + aq_{xx} + 2b\ln|q|q = i\left[\alpha q_x + \nu\left(|q|^{2m}\right)_x q\right] \quad (90)$$

and Eq.(89) reduces to

$$a\left(\left(V'\right)^2 + V''\right) - \omega - \alpha\kappa - a\kappa^2 + 2bV = 0 \quad (91)$$

Balancing $\left(V'\right)^2$ with V in Eq.(91), then we get $N = 1$. Using the solution procedure of the trial equation method, we obtain the system of algebraic equations as follows:

V^1 Coeff.:

$$aa_1 + 2b = 0,$$

V^0 Coeff.:

$$\frac{aa_1}{2} + aa_0 - \omega - \alpha\kappa - a\kappa^2 = 0.$$

Solving the above system leads to

$$a_0 = \frac{b + \omega + \alpha\kappa + a\kappa^2}{a}, \quad a_1 = \frac{-2b}{a}$$

where ω, α, κ, a, b are arbitrary constants.

Substituting these results into Eqs.(5) and (6), we get

$$\pm\left(\xi - \xi_0\right) = \int \frac{dV}{\sqrt{\frac{b + \omega + \alpha\kappa + a\kappa^2}{a} - \frac{2b}{a}V}} \quad (92)$$

Integrating Eq.(92), we obtain the exact Gausson solutions of Eq.(90) as

$$q(x,t) = Ae^{-B^2(x-vt)^2} e^{i(-\kappa x + \omega t + \theta)} \quad (93)$$

where the amplitude A and the inverse width B are

$$A = \exp\left(\frac{b + \omega + \alpha\kappa + a\kappa^2}{2b}\right) \quad (94)$$

and

$$B = \sqrt{\frac{b}{2a}} \quad (95)$$

Naturally, the width of the Gausson proposes the restriction

$$ab > 0.$$

This shows that the nonlinearity and GVD must bear the same sign for the existence of Gaussons.

3.7. Anti-Cubic Law

This anti-cubic (AC) law first appeared during 2003 ([8], [9], [11], [14]). Later, a lot of developments were made and results were reported. In this case,

$$F(s) = \frac{b_7}{s^2} + b_8 s + b_9 s^2$$

where b_7, b_8 and b_9 are all constants. Therefore NLSE with AC nonlinearity is given by

$$iq_t + aq_{xx} + \left(b_7|q|^{-4} + b_8|q|^2 + b_9|q|^4\right)q = i\left[\alpha q_x + \lambda\left(|q|^{2m} q\right)_x + \nu\left(|q|^{2m}\right)_x q\right]. \quad (96)$$

In this case, Eq.(9) simplifies to

$$aU'' - \left(\omega + \alpha\kappa + a\kappa^2\right)U - \lambda\kappa U^{2m+1} + b_7 U^{-3} + b_8 U^3 + b_9 U^5 = 0 \quad (97)$$

By using transformation $U = V^{\frac{1}{2}}$, Eq.(97) becomes

$$a\left(-\left(V'\right)^2 + 2VV''\right) - 4V^2\left(\omega + \alpha\kappa + a\kappa^2\right) - 4\lambda\kappa V^{m+2} + 4\left(b_7 + b_8 V^3 + b_9 V^4\right) = 0 \quad (98)$$

Balancing VV'' or $\left(V'\right)^2$ with V^{m+2} in Eq.(98), then we get $N = m + 2$. Using the solution procedure of the trial equation method, we obtain the system of algebraic equations as follows:

V^{m+2} Coeff.:

$$-4\lambda\kappa + a(m+1)a_{m+2} = 0,$$

V^4 Coeff.:

$$4b_9 + 3aa_4 = 0,$$

V^3 Coeff.:

$$4b_8 + 2aa_3 = 0,$$

V^2 Coeff.:

$$-4\left(\omega + \alpha\kappa + a\kappa^2\right) + aa_2 = 0,$$

V^0 Coeff.:

$$-aa_0 + 4b_7 = 0.$$

Solving the above system leads to

$$a_0 = \frac{4b_7}{a},\ a_2 = \frac{4\left(\omega + \alpha\kappa + a\kappa^2\right)}{a},$$
$$a_3 = -\frac{2b_8}{a},\ \ a_4 = -\frac{4b_9}{3a},\ \ a_{m+2} = \frac{4\lambda\kappa}{a(m+1)}$$

and also apparent coefficients from Eq.(98)

$$a_5 = a_6 = ... = a_{m+1} = 0$$

where a_1, ω, α, κ, a, b_7, b_8, b_9, λ are arbitrary constants.

Substituting these results into Eqs.(5) and (6), we get

$$\pm\left(\xi - \xi_0\right) = \int \frac{dV}{\sqrt{\frac{4b_7}{a} + a_1 V + \frac{4\left(\omega + \alpha\kappa + a\kappa^2\right)}{a} V^2 - \frac{2b_8}{a} V^3 - \frac{4b_9}{3a} V^4 + \frac{4\lambda\kappa}{a(m+1)} V^{m+2}}} \tag{99}$$

To carry out the integration of Eq.(99) requires that $m = 1$. Thus, with $m = 1$, new Eq.(99) is following:

$$\pm\left(\xi - \xi_0\right) = \int \frac{dV}{\sqrt{\frac{4b_7}{a} + a_1 V + \frac{4\left(\omega + \alpha\kappa + a\kappa^2\right)}{a} V^2 + \frac{2\left(\lambda\kappa - b_8\right)}{a} V^3 - \frac{4b_9}{3a} V^4}} \tag{100}$$

Eq.(100) can be integrated with respect to V if we set

$$a_1 = 0,\ b_8 = \lambda\kappa,\ b_7 = -\frac{3\left(\omega + \alpha\kappa + a\kappa^2\right)^2}{4b_9}$$

Thus, we obtain exact solutions of Eq.(96):

$$q(x,t) = \pm\left\{\sqrt{-\frac{3\left(\omega + \alpha\kappa + a\kappa^2\right)}{2b_9}} \tan\left[\sqrt{\frac{2\left(\omega + \alpha\kappa + a\kappa^2\right)}{a}} (x - vt)\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x + \omega t + \theta)}, \tag{101}$$

$$q(x,t) = \pm\left\{\sqrt{-\frac{3\left(\omega + \alpha\kappa + a\kappa^2\right)}{2b_9}} \cot\left[\sqrt{\frac{2\left(\omega + \alpha\kappa + a\kappa^2\right)}{a}} (x - vt)\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x + \omega t + \theta)}, \tag{102}$$

where Eq.(101) and Eq.(102) represent singular periodic solutions. These solutions are valid for

$$a\left(\omega + \alpha\kappa + a\kappa^2\right) > 0.$$

$$q(x,t) = \pm\left\{\sqrt{\frac{3\left(\omega + \alpha\kappa + a\kappa^2\right)}{2b_9}} \tanh\left[\sqrt{-\frac{2\left(\omega + \alpha\kappa + a\kappa^2\right)}{a}} (x - vt)\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x + \omega t + \theta)}, \tag{103}$$

$$q(x,t) = \pm\left\{\sqrt{\frac{3\left(\omega + \alpha\kappa + a\kappa^2\right)}{2b_9}} \coth\left[\sqrt{-\frac{2\left(\omega + \alpha\kappa + a\kappa^2\right)}{a}} (x - vt)\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x + \omega t + \theta)}, \tag{104}$$

where Eq.(103) and Eq.(104) represent dark and singular soliton solution respectively. These solitons are valid for

$$a\left(\omega+\alpha\kappa+a\kappa^2\right)<0.$$

3.8. Cubic-Quintic-Septic Law

This is an extension of parabolic law nonlinearity ([5], [6], [17]). This type of nonlinear optical medium takes the form

$$F(s)=b_{10}s+b_{11}s^2+b_{12}s^3$$

where b_{10}, b_{11} and b_{12} are all constants. Therefore NLSE is given by

$$iq_t+aq_{xx}+\left(b_{10}|q|^2+b_{11}|q|^4+b_{12}|q|^6\right)q= i\left[\alpha q_x+\lambda\left(|q|^{2m}q\right)_x+\nu\left(|q|^{2m}\right)_x q\right] \tag{105}$$

In this case, Eq.(9) simplifies to

$$aU''-\left(\omega+\alpha\kappa+a\kappa^2\right)U-\lambda\kappa U^{2m+1}+ b_{10}U^3+b_{11}U^5+b_{12}U^7=0 \tag{106}$$

By using transformation $U=V^{\frac{1}{2}}$, Eq.(106) becomes

$$a\left(-\left(V'\right)^2+2VV''\right)-4V^2\left(\omega+\alpha\kappa+a\kappa^2\right)- 4\lambda\kappa V^{m+2}+4b_{10}V^3+4b_{11}V^4+4b_{12}V^5=0 \tag{107}$$

Balancing VV'' or $\left(V'\right)^2$ with V^{m+2} in Eq.(107), then we get $N=m+2$. Using the solution procedure of the trial equation method, we obtain the system of algebraic equations as follows:

V^{m+2} Coeff.:

$$-4\lambda\kappa+a(m+1)a_{m+2}=0,$$

V^5 Coeff.:

$$b_{12}+aa_5=0,$$

V^4 Coeff.:

$$4b_{11}+3aa_4=0,$$

V^3 Coeff.:

$$2b_{10}+aa_3=0,$$

V^2 Coeff.:

$$-4\left(\omega+\alpha\kappa+a\kappa^2\right)+aa_2=0,$$

V^0 Coeff.:

$$-aa_0=0.$$

Solving the above system leads to

$$a_0=0,\ a_2=\frac{4\left(\omega+\alpha\kappa+a\kappa^2\right)}{a},\ a_3=-\frac{2b_{10}}{a},$$

$$a_4=-\frac{4b_{11}}{3a},a_5=-\frac{b_{12}}{a},\ a_{m+2}=\frac{4\lambda\kappa}{a(m+1)}$$

and also apparent coefficients from Eq.(107)

$$a_6=a_7=\ldots=a_{m+1}=0$$

where a_1, ω, α, κ, a, b_{10}, b_{11}, b_{12}, λ are arbitrary constants.
we get

$$\pm\left(\xi-\xi_0\right)= \int\frac{dV}{\sqrt{a_1V+\frac{4\left(\omega+\alpha\kappa+a\kappa^2\right)}{a}V^2-\frac{2b_{10}}{a}V^3-\frac{4b_{11}}{3a}V^4-\frac{b_{12}}{a}V^5+\frac{4\lambda\kappa}{a(m+1)}V^{m+2}}} \tag{108}$$

To carry out the integration of Eq.(108) requires that $m=3$. Thus, with $m=3$, new Eq.(108) is following:

$$\pm\left(\xi-\xi_0\right)= \int\frac{dV}{\sqrt{a_1V+\frac{4\left(\omega+\alpha\kappa+a\kappa^2\right)}{a}V^2-\frac{2b_{10}}{a}V^3-\frac{4b_{11}}{3a}V^4+\frac{\left(\lambda\kappa-b_{12}\right)}{a}V^5}} \tag{109}$$

Eq.(109) can be integrated with respect to V if we set

$$a_1=0, b_{12}=\lambda\kappa, b_{11}=-\frac{3b_{10}^2}{16\left(\omega+\alpha\kappa+a\kappa^2\right)}$$

Thus, we obtain exact solutions of Eq.(105):

$$q(x,t)=\sqrt{\frac{2\left(\omega+\alpha\kappa+a\kappa^2\right)}{b_{10}}} \left\{1\pm\tanh\left[\sqrt{\frac{\left(\omega+\alpha\kappa+a\kappa^2\right)}{a}}\left(x-vt\right)\right]\right\}^{\frac{1}{2}} e^{i(-\kappa x+\omega t+\theta)}, \tag{110}$$

$$q(x,t)=\sqrt{\frac{2\left(\omega+\alpha\kappa+a\kappa^2\right)}{b_{10}}}\left\{1\pm\coth\left[\sqrt{\frac{\left(\omega+\alpha\kappa+a\kappa^2\right)}{a}}(x-vt)\right]\right\}^{\frac{1}{2}}e^{i(-\kappa x+\omega t+\theta)}, \quad (111)$$

where Eq.(110) and Eq.(111) represent dark and singular soliton solution respectively. These solitons are valid for

$$a\left(\omega+\alpha\kappa+a\kappa^2\right)>0.$$

3.9. Triple-Power Law

This is a generalization of cubic-quintic-septic law nonlinearity and an extension of dual-power law nonlinearity ([5], [6], [17]). In this case,

$$F(s)=b_{13}s^n+b_{14}s^{2n}+b_{15}s^{3n}$$

where b_{13}, b_{14} and b_{15} are all constants. Therefore NLSE is given by

$$iq_t+aq_{xx}+\left(b_{13}|q|^{2n}+b_{14}|q|^{4n}+b_{15}|q|^{6n}\right)q = i\left[\alpha q_x+\lambda\left(|q|^{2m}q\right)_x+\nu\left(|q|^{2m}\right)_x q\right] \quad (112)$$

In this case, Eq.(9) simplifies to

$$aU''-\left(\omega+\alpha\kappa+a\kappa^2\right)U-\lambda\kappa U^{2m+1}+b_{13}U^{2n+1}+b_{14}U^{4n+1}+b_{15}U^{6n+1}=0 \quad (113)$$

By using transformation $U=V^{\frac{1}{2n}}$, Eq.(113) becomes

$$a\left((1-2n)\left(V'\right)^2+2nVV''\right)-4n^2V^2\left(\omega+\alpha\kappa+a\kappa^2\right)-4n^2\lambda\kappa V^{\frac{m}{n}+2}+4n^2b_{13}V^3+4n^2b_{14}V^4+4n^2b_{15}V^5=0 \quad (114)$$

Balancing VV'' or $\left(V'\right)^2$ with V^{m+2} in Eq.(114), then we get $N=\frac{m}{n}+2$. Using the solution procedure of the trial equation method, we obtain the system of algebraic equations as follows:

V^{m+2} Coeff.:

$$-4n^2\lambda\kappa+a(m+1)a_{\frac{m}{n}+2}=0,$$

V^5 Coeff.:

$$4n^2b_{15}+a(1+3n)a_5=0,$$

V^4 Coeff.:

$$4n^2b_{14}+a(1+2n)a_4=0,$$

V^3 Coeff.:

$$4n^2b_{13}+a(1+n)a_3=0,$$

V^2 Coeff.:

$$-4n^2\left(\omega+\alpha\kappa+a\kappa^2\right)+aa_2=0,$$

V^1 Coeff.:

$$a(1-n)a_1=0,$$

V^0 Coeff.:

$$a(1-2n)a_0=0.$$

Solving the above system leads to

$$a_0=0,\ a_1=0,\ a_2=\frac{4n^2\left(\omega+\alpha\kappa+a\kappa^2\right)}{a},$$

$$a_3=-\frac{4n^2b_{13}}{a(1+n)},a_4=-\frac{4n^2b_{14}}{a(1+2n)}$$

$$,\ a_5=-\frac{4n^2b_{15}}{a(1+3n)},\ a_{\frac{m}{n}+2}=\frac{4n^2\lambda\kappa}{a(m+1)}$$

and also apparent coefficients from Eq.(114)

$$a_6=a_7=...=a_{\frac{m}{n}+1}=0$$

where ω, α, κ, a, b_{13}, b_{14}, b_{15}, λ are arbitrary constants.

we get

$$\pm(\xi-\xi_0)=\int\frac{dV}{\sqrt{\frac{4n^2\left(\omega+\alpha\kappa+a\kappa^2\right)}{a}V^2-\frac{4n^2b_{13}}{a(1+n)}V^3-\frac{4n^2b_{14}}{a(1+2n)}V^4-\frac{4n^2b_{15}}{a(1+3n)}V^5+\frac{4n^2\lambda\kappa}{a(m+1)}V^{\frac{m}{n}+2}}} \quad (115)$$

To carry out the integration of Eq.(115) requires that $m=3n$. Thus, with $m=3n$, new Eq.(115) is following:

$$\pm(\xi-\xi_0)=\int\frac{dV}{\sqrt{\frac{4n^2(\omega+\alpha\kappa+a\kappa^2)}{a}V^2-\frac{4n^2b_{13}}{a(1+n)}V^3-\frac{4n^2b_{14}}{a(1+2n)}V^4+\frac{4n^2(\lambda\kappa-b_{15})}{a(1+3n)}V^5}} \quad (116)$$

Eq.(116) can be integrated with respect to V if we set

$$b_{15}=\lambda\kappa,\ b_{14}=-\frac{(1+2n)b_{13}^2}{4(1+n)^2(\omega+\alpha\kappa+a\kappa^2)} \quad (117)$$

Thus, we obtain exact solutions of Eq.(112

$$q(x,t)=\left\{\frac{(1+n)(\omega+\alpha\kappa+a\kappa^2)}{b_{13}}\left[1\pm\tanh\left(\sqrt{\frac{n^2(\omega+\alpha\kappa+a\kappa^2)}{a}}(x-vt)\right)\right]\right\}^{\frac{1}{2n}} e^{i(-\kappa x+\omega t+\theta)}, \quad (118)$$

$$q(x,t)=\left\{\frac{(1+n)(\omega+\alpha\kappa+a\kappa^2)}{b_{13}}\left[1\pm\coth\left(\sqrt{\frac{n^2(\omega+\alpha\kappa+a\kappa^2)}{a}}(x-vt)\right)\right]\right\}^{\frac{1}{2n}} e^{i(-\kappa x+\omega t+\theta)}, \quad (119)$$

where Eq.(118) and Eq.(119) represent dark and singular soliton respectively and they are valid for

$$a(\omega+\alpha\kappa+a\kappa^2)>0. \quad (120)$$

4. Conclusions

In this paper, optical solitons are studied in the presence of perturbed nonlinear Schrödinger's equation with nine types of nonlinear fibers which are Kerr, power, quadratic-cubic, parabolic, dual-power, log, anti-cubic, cubic-quintic-septic and triple-power law nonlinearity. Bright, dark and singular solitons are yielded by the trial equation method along with necessary constraint conditions that guaranteed the existence of such solitons. On the flip side, singular periodic solutions emerged with reverse form of the constraints.

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