Journal of Nonlinear Optical Physics & Materials
Vol. 24, No. 2 (2015) 1550017 (41 pages)

DOI: 10.1142/S0218863515500174

Optical solitons in nonlinear directional couplers with G'/G-expansion scheme

Mohammad Mirzazadeh*, Mostafa Eslami†, Qin Zhou‡,
M. F. Mahmood§, Essaid Zerrad¶, Anjan Biswas‖,** and Milivoj Belic††

*Department of Mathematics, Faculty of Mathematical Sciences,
University of Guilan, Rasht, Iran

†Department of Mathematics, Faculty of Mathematical Sciences,
University of Mazandaran, Babolsar, Iran

‡School of Electronics and Information Engineering,
Wuhan Donghu University, Wuhan 430212, People's Republic of China

§Department of Mathematics, Howard University,
Washington, DC 20059, USA

¶Department of Physics and Engineering,
Delaware State University, Dover, Delaware 19901-2277, USA

‖Department of Mathematical Sciences,
Delaware State University, Dover, Delaware 19901-2277, USA

**Department of Mathematics, Faculty of Science,
King Abdulaziz University, Jeddah 21589, Saudi Arabia

††Science Program, Texas A and M University at Qatar,
P. O. Box 23874, Doha, Qatar

**biswas.anjan@gmail.com

Received 4 March 2015

This paper obtains soliton solutions in optical couplers. The governing equation is solved by the aid of G'/G-expansion scheme. There are four types of nonlinear media that are taken into consideration. These are Kerr law, power law, parabolic law, and dual-power law. There are two kinds optical couplers studied in this paper. They are twin-core couplers and multiple-core couplers, where coupling with nearest neighbors as well as coupling with all neighbors are considered. Dark and singular soliton solutions are retrieved. These soliton solutions come with constraint conditions that must hold for the solitons to exist.

Keywords: Solitons; couplers; integrability.

1. Introduction

Optical solitons is the basic fabric of fiber-optic communications across trans-continental and trans-oceanic distances. These soliton molecules are studied in several different context in this field.[1–40] They are optical fibers, optical couplers, optical switching, and several others. This paper will focus on obtaining soliton solutions in optical couplers. The governing equation will be the nonlinear Schrödinger's equation (NLSE). This paper will consider NLSE with spatio-temporal dispersion (STD) in addition to the usual group velocity dispersion (GVD). The inclusion of STD makes the governing equation well-posed as pointed out during 2012.[13,21] There are dark and singular solitons that will be retrieved from the model. The four forms of nonlinear media that will be studied in this paper are Kerr law, power law, parabolic law and dual-power law. The case of log-law nonlinearity with Gaussons was reported in the past.[8,38] Several constraint conditions will be revealed that will guarantee the existence of these soliton solutions.

There are several integrability issues that exist in the literature of partial differential equations (PDEs). Several papers have been reported on optical couplers where bright, dark and singular soliton solutions are reported.[1–8,38] This paper adopts a different integrability criteria. This is G'/G-expansion scheme. The limitation of this approach of integration is that this scheme only retrieves dark and singular solitons only. The bright soliton solutions cannot be recovered with this algorithm. However, such soliton solutions are retrievable using other scheme such as ansatz method that was already reported during 2014.[38] The following section gives a mathematical overview of G'/G-expansion scheme. Subsequently, this will be implemented to recover soliton solutions. There are two types of couplers that will be studied in this paper. They are twin-core couplers and multiple-core couplers.

2. Recapitulation of G'/G-Expansion Scheme

We suppose that the given nonlinear evolution equation for $u(x, t)$ is in the form

$$P(u, u_t, u_x, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0, \tag{1}$$

where P is a polynomial. The essence of the G'/G-expansion method can be presented in the following steps[21,29]:

Step 1: To find the traveling wave solutions of Eq. (1), we introduce the wave variable

$$u(x, t) = U(\tau), \quad \tau = B(x - vt). \tag{2}$$

Substituting Eq. (2) into Eq. (1), we obtain the following ordinary differential equation (ODE)

$$Q(U, U', U'', \ldots) = 0. \tag{3}$$

Step 2: Equation (3) is then integrated as long as all terms contain derivatives where integration constants are considered zeros.

Step 3: Introduce the solution $U(\tau)$ of Eq. (3) in the finite series form

$$U(\tau) = \sum_{l=0}^{N} A_l \left(\frac{G'(\tau)}{G(\tau)}\right)^l, \tag{4}$$

where A_l are real constants with $A_N \neq 0$ and N is a positive integer to be determined. The function $G(\tau)$ is the solution of the auxiliary linear ordinary differential equation (LODE)

$$G''(\tau) + \lambda G'(\tau) + \mu G(\tau) = 0, \tag{5}$$

where λ and μ are real constants to be determined.

Step 4: Determining N can be accomplished by balancing the linear term of highest order derivatives with the highest order nonlinear term in Eq. (3).

Step 5: Substituting the general solution of (5) together with (4) into Eq. (3) yields an algebraic equation involving powers of G'/G. Equating the coefficients of each power of G'/G to zero gives a system of algebraic equations for A_l, λ, μ and c. Then, we solve the system with the aid of a computer algebra system, such as *Maple*, to determine these constants. Next, depending on the sign of the discriminant $\Delta = \lambda^2 - 4\mu$, we get solutions of Eq. (3). So, we can obtain exact solutions of Eq. (1).

3. Twin-Core Couplers

The governing equation for twin-core couplers is given by[1−6,38]

$$iq_t + a_1 q_{xx} + b_1 q_{xt} + c_1 F(|q|^2)q = k_1 r, \tag{6}$$

$$ir_t + a_2 r_{xx} + b_2 r_{xt} + c_2 F(|r|^2)r = k_2 q. \tag{7}$$

Equations (6) and (7) represent the coupled NLSE, with GVD and STD, that governs soliton propagation through twin-core optical fibers, typically for non-Kerr law media. The first term, for both equations, represents the evolution term. The coefficients of GVD are a_l, while the coefficients of STD are b_l for $l = 1, 2$. Then c_l represents the coefficients of nonlinearity where the functional F gives the type of nonlinearity that will be studied. Here, $F(|q|^2)q : C \mapsto C$. Considering the complex plane C as a two-dimensional linear space R_2, the function $F(|q|^2)q$ is k times continuously differentiable, so that

$$F(|q|^2)q \in \bigcup_{m,n=1}^{\infty} C^k((-n, n) \times (-m, m); R^2). \tag{8}$$

On the right-hand sides of (6) and (7), constants k_1 and k_2 represent the coupling coefficients. In order to study integrability of these equations by G'/G-expansion scheme, the following solution structure is selected.

$$q(x, t) = P_1(x, t)e^{i\phi(x,t)}, \tag{9}$$

$$r(x, t) = P_2(x, t)e^{i\phi(x,t)}, \tag{10}$$

where $P_l(x, t)$ $(l = 1, 2)$ represents the amplitude component of the soliton solution while the phase component $\phi(x, t)$ is defined as

$$\phi(x, t) = -\kappa x + \omega t + \theta. \tag{11}$$

Here κ is the frequency of the solitons while ω represents the wave number and θ is the phase constant. From Eq. (11), it is clear that the phase for both the couplers are same and this is referred to as "phase-matching" condition. This condition is necessary for integrability purpose. Without this condition, the integrability aspect fails.

Substituting (9) and (10) into (6) and (7) and then decomposing into real and imaginary parts gives

$$a_l \frac{\partial^2 P_l}{\partial x^2} + b_l \frac{\partial^2 P_l}{\partial x \partial t} + P_l(b_l \omega \kappa - \omega - a_l \kappa^2) + c_l F(P_l^2) P_l - k_l P_{\bar{l}} = 0 \tag{12}$$

and

$$(1 - b_l \kappa) \frac{\partial P_l}{\partial t} + (b_l \omega - 2a_l \kappa) \frac{\partial P_l}{\partial x} = 0, \tag{13}$$

respectively. Here, $l = 1, 2$ and $\bar{l} = 3 - l$. Under the travelling wave transformation

$$P_1(x,t) = U_1(\tau), \quad P_2(x,t) = U_2(\tau), \quad \tau = B(x - vt) \tag{14}$$

we have

$$(a_l - b_l v) B^2 \frac{d^2 U_l}{d\tau^2} + U_l(b_l \omega \kappa - \omega - a_l \kappa^2) + c_l F(U_l^2) U_l - k_l U_{\bar{l}} = 0 \tag{15}$$

and

$$(-v(1 - b_l \kappa) + b_l \omega - 2a_l \kappa) B \frac{dU_l}{d\tau} = 0. \tag{16}$$

Now, from Eq. (16), we get

$$v = \frac{b_l \omega - 2a_l \kappa}{1 - b_l \kappa}. \tag{17}$$

Now, equating the two values of the soliton speed leads to

$$a_1 = a_2 \tag{18}$$

and

$$b_1 = b_2. \tag{19}$$

The speed of the soliton therefore reduces to

$$v = \frac{b\omega - 2a\kappa}{1 - b\kappa}. \tag{20}$$

The coupled NLSE for twin-core couplers given by (6) and (7) modifies to

$$iq_t + aq_{xx} + bq_{xt} + c_1 F(|q|^2) q = k_1 r, \tag{21}$$

$$ir_t + ar_{xx} + br_{xt} + c_2 F(|r|^2) r = k_2 q, \tag{22}$$

where $a_1 = a_2 = a$ and $b_1 = b_2 = b$. Consequently, Eq. (15) changes to

$$(a - bv) B^2 \frac{d^2 U_l}{d\tau^2} + U_l(b\omega \kappa - \omega - a\kappa^2) + c_l F(U_l^2) U_l - k_l U_{\bar{l}} = 0. \tag{23}$$

3.1. *Kerr law nonlinearity*

For Kerr law nonlinearity, $F(s) = s$. The model equations (21) and (22), for twin-core couplers with Kerr law nonlinearity, reduces to

$$iq_t + aq_{xx} + bq_{xt} + c_1|q|^2 q = k_1 r, \tag{24}$$

$$ir_t + ar_{xx} + br_{xt} + c_2|r|^2 r = k_2 q. \tag{25}$$

Therefore, real part equation (23) is

$$(a - bv)B^2 \frac{d^2U_1}{d\tau^2} + U_1(b\omega\kappa - \omega - a\kappa^2) + c_1 U_1^3 - k_1 U_2 = 0, \tag{26}$$

$$(a - bv)B^2 \frac{d^2U_2}{d\tau^2} + U_2(b\omega\kappa - \omega - a\kappa^2) + c_2 U_2^3 - k_2 U_1 = 0. \tag{27}$$

According to the homogeneous balance method, Eqs. (26) and (27) have the solutions in the form

$$U_1(\tau) = A_0 + A_1\left(\frac{G'(\tau)}{G(\tau)}\right), \tag{28}$$

$$U_2(\tau) = B_0 + B_1\left(\frac{G'(\tau)}{G(\tau)}\right), \tag{29}$$

where $G(\tau)$ satisfies the second-order LODE

$$G''(\tau) + \lambda G'(\tau) + \mu G(\tau) = 0, \tag{30}$$

where λ and μ are real constants to be determined.

Substituting Eqs. (28) and (29) into Eqs. (26) and (27) leads to

$$\begin{aligned}(a - bv)B^2&\left\{2A_1\left(\frac{G'}{G}\right)^3 + 3A_1\lambda\left(\frac{G'}{G}\right)^2 + \left(2A_1\mu + A_1\lambda^2\right)\left(\frac{G'}{G}\right) + \lambda\mu A_1\right\}\\ &+ (b\omega\kappa - \omega - a\kappa^2)\left\{A_0 + A_1\left(\frac{G'}{G}\right)\right\} + c_1\left\{A_0 + A_1\left(\frac{G'}{G}\right)\right\}^3\\ &- k_1\left\{B_0 + B_1\left(\frac{G'}{G}\right)\right\} = 0,\end{aligned} \tag{31}$$

$$\begin{aligned}(a - bv)B^2&\left\{2B_1\left(\frac{G'}{G}\right)^3 + 3B_1\lambda\left(\frac{G'}{G}\right)^2 + (2B_1\mu + B_1\lambda^2)\left(\frac{G'}{G}\right) + \lambda\mu B_1\right\}\\ &+ (b\omega\kappa - \omega - a\kappa^2)\left\{B_0 + B_1\left(\frac{G'}{G}\right)\right\} + c_2\left\{B_0 + B_1\left(\frac{G'}{G}\right)\right\}^3\\ &- k_2\left\{A_0 + A_1\left(\frac{G'}{G}\right)\right\} = 0.\end{aligned} \tag{32}$$

Then, equating the coefficient of each power of G'/G to zero, we obtain a system of nonlinear algebraic equations and by solving it, we get

$$A_0 = \pm\frac{\lambda}{2}\sqrt{\frac{c_2}{c_1}}B_1, \quad A_1 = \pm\sqrt{\frac{c_2}{c_1}}B_1, \tag{33}$$

$$B_0 = \frac{\lambda}{2}B_1, \quad v = \frac{c_2 B_1^2 + 2B^2 a}{2bB^2}, \quad k_2 = \frac{k_1 c_1}{c_2}, \tag{34}$$

$$\omega = \frac{B_1^2 c_2^2(4\mu - \lambda^2) + 4a\kappa^2 c_2 \pm 4k_1\sqrt{c_1 c_2}}{4c_2(b\kappa - 1)}, \tag{35}$$

where $B,\ B_1\ \kappa,\ \lambda,\ \mu$ are arbitrary constants.

Equating the two expressions for the soliton speed v from (20) and (34) implies

$$B = \pm\sqrt{\frac{(1 - b\kappa)c_2}{2(b^2\omega - a\kappa b - a)}}B_1, \tag{36}$$

which immediately prompts the constraint

$$(1 - b\kappa)c_2(b^2\omega - a\kappa b - a) > 0. \tag{37}$$

Finally, equating the two components of the soliton width B gives the ratio of the soliton amplitudes as

$$\frac{A_l}{B_l} = \sqrt{\frac{c_2}{c_1}}, \quad l = 0, 1, \tag{38}$$

which immediately prompts the constraint

$$c_1 c_2 > 0. \tag{39}$$

Substituting the solution set (33)–(35) into Eqs. (28) and (29), the solution formulae of Eqs. (26) and (27) can be written as

$$U_1(\tau) = \pm\sqrt{\frac{c_2}{c_1}}B_1\left\{\frac{\lambda}{2} + \frac{G'(\tau)}{G(\tau)}\right\}, \tag{40}$$

$$U_2(\tau) = B_1\left\{\frac{\lambda}{2} + \frac{G'(\tau)}{G(\tau)}\right\}. \tag{41}$$

Substituting the general solutions of second-order linear ODE into Eqs. (40) and (41) gives three types of traveling wave solutions.

When $\Delta = \lambda^2 - 4\mu > 0$, the hyperbolic function traveling wave solutions are:

$$q(x,t) = \pm\sqrt{\frac{c_2(\lambda^2 - 4\mu)}{4c_1}}B_1 \times \left\{\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}B(x - vt)\right) + C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}B(x - vt)\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}B(x - vt)\right) + C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}B(x - vt)\right)}\right\} \times e^{i(-\kappa x + \omega t + \theta)}, \tag{42}$$

$$r(x,t) = \sqrt{\frac{\lambda^2-4\mu}{4}} B_1 \times \left\{ \frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2} B(x-vt)\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2} B(x-vt)\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2} B(x-vt)\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2} B(x-vt)\right)} \right\} \times e^{i(-\kappa x+\omega t+\theta)}, \quad (43)$$

where C_1 and C_2 are arbitrary constants and B is given by Eq. (36), while v is given by Eq. (20), and ω is given by Eq. (35). There are three constraint conditions in order for these analytical solutions to exist, they are given by Eqs. (37) and (39) and $k_2 = (k_1 c_1)/c_2$, respectively.

On the other hand, assuming $C_1 \neq 0$ and $C_2 = 0$, dark 1-soliton solutions of Eqs. (24) and (25) can be written as:

$$q(x,t) = \pm\sqrt{\frac{c_2(\lambda^2-4\mu)}{4c_1}} B_1 \tanh\left[\frac{\sqrt{\lambda^2-4\mu}}{2} B(x-vt)\right] e^{i(-\kappa x+\omega t+\theta)}, \quad (44)$$

$$r(x,t) = \sqrt{\frac{\lambda^2-4\mu}{4}} B_1 \tanh\left[\frac{\sqrt{\lambda^2-4\mu}}{2} B(x-vt)\right] e^{i(-\kappa x+\omega t+\theta)}. \quad (45)$$

Next, assuming $C_1 = 0$ and $C_2 \neq 0$, then we obtain singular 1-soliton solution for Eqs. (24) and (25) as

$$q(x,t) = \pm\sqrt{\frac{c_2(\lambda^2-4\mu)}{4c_1}} B_1 \coth\left[\frac{\sqrt{\lambda^2-4\mu}}{2} B(x-vt)\right] e^{i(-\kappa x+\omega t+\theta)}, \quad (46)$$

$$r(x,t) = \sqrt{\frac{\lambda^2-4\mu}{4}} B_1 \coth\left[\frac{\sqrt{\lambda^2-4\mu}}{2} B(x-vt)\right] e^{i(-\kappa x+\omega t+\theta)}, \quad (47)$$

where B is given by Eq. (36), while v is given by Eq. (20), and ω is given by Eq. (35). There are three constraint conditions in order for these analytical solutions to exist, they are given by Eqs. (37) and (39) and $k_2 = (k_1 c_1)/c_2$, respectively.

However, when $\Delta = \lambda^2 - 4\mu < 0$, we obtain trig function traveling wave solutions:

$$q(x,t) = \pm\sqrt{\frac{c_2(4\mu-\lambda^2)}{4c_1}} B_1 \times \left\{ \frac{-C_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2} B(x-vt)\right) + C_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2} B(x-vt)\right)}{C_1 \cos\left(\frac{\sqrt{\lambda^2-4\mu}}{2} B(x-vt)\right) + C_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2} B(x-vt)\right)} \right\} \times e^{i(-\kappa x+\omega t+\theta)}, \quad (48)$$

$$r(x,t) = \sqrt{\frac{4\mu-\lambda^2}{4}} B_1 \times \left\{ \frac{-C_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2} B(x-vt)\right) + C_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2} B(x-vt)\right)}{C_1 \cos\left(\frac{\sqrt{\lambda^2-4\mu}}{2} B(x-vt)\right) + C_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2} B(x-vt)\right)} \right\} \times e^{i(-\kappa x+\omega t+\theta)}, \tag{49}$$

where C_1 and C_2 are arbitrary constants.

Also, with the assumption $C_1 \neq 0$ and $C_2 = 0$, gives

$$q(x,t) = \mp\sqrt{\frac{c_2(4\mu-\lambda^2)}{4c_1}} B_1 \tan\left[\frac{\sqrt{4\mu-\lambda^2}}{2} B(x-vt)\right] e^{i(-\kappa x+\omega t+\theta)}, \tag{50}$$

$$r(x,t) = -\sqrt{\frac{4\mu-\lambda^2}{4}} B_1 \tan\left[\frac{\sqrt{4\mu-\lambda^2}}{2} B(x-vt)\right] e^{i(-\kappa x+\omega t+\theta)}, \tag{51}$$

while the assumption $C_1 = 0, C_2 \neq 0$ one recovers

$$q(x,t) = \pm\sqrt{\frac{c_2(4\mu-\lambda^2)}{4c_1}} B_1 \cot\left[\frac{\sqrt{4\mu-\lambda^2}}{2} B(x-vt)\right] e^{i(-\kappa x+\omega t+\theta)}, \tag{52}$$

$$r(x,t) = \sqrt{\frac{4\mu-\lambda^2}{4}} B_1 \cot\left[\frac{\sqrt{4\mu-\lambda^2}}{2} B(x-vt)\right] e^{i(-\kappa x+\omega t+\theta)}, \tag{53}$$

where B is given by Eq. (36), while v is given by Eq. (20), and ω is given by Eq. (35). These solutions given by (50)–(53) are commonly referred to singular-periodic solutions. There are three constraint conditions in order for these analytical solutions to exist. They are given by Eqs. (37) and (39) and $k_2 = (k_1c_1)/c_2$, respectively.

When $\Delta = \lambda^2 - 4\mu = 0$, the plane wave solutions are:

$$q(x,t) = \pm\sqrt{\frac{c_2}{c_1}} \frac{B_1 C_2}{C_1 + C_2 B(x-vt)} e^{i(-\kappa x+\omega t+\theta)} \tag{54}$$

and

$$r(x,t) = \frac{B_1 C_2}{C_1 + C_2 B(x-vt)} e^{i(-\kappa x+\omega t+\theta)}, \tag{55}$$

where C_1 and C_2 are arbitrary constants and B is given by Eq. (36), while v is given by Eq. (20), and ω is given by Eq. (35). There are three constraint conditions in order for these analytical solutions to exist, they are given by Eqs. (37) and (39) and $k_2 = (k_1c_1)/c_2$, respectively.

3.2. *Power law nonlinearity*

For power law nonlinear media, $F(s) = s^n$ where n represents the power law non-linearity factor. Thus for twin-core couplers, the NLSE[1–6,38]

$$iq_t + aq_{xx} + bq_{xt} + c|q|^{2n}q = k_1 r, \tag{56}$$

$$ir_t + ar_{xx} + br_{xt} + c_2|r|^{2n}r = k_2 q. \tag{57}$$

It must be noted that $0 < n < 2$ for stability of solitons. Additionally $n \neq 2$ for self-focusing singularity. Therefore, real part Eq. (23) is

$$(a - bv)B^2 \frac{d^2U_1}{d\tau^2} + U_1(b\omega\kappa - \omega - a\kappa^2) + c_1 U_1^{2n+1} - k_1 U_2 = 0, \tag{58}$$

$$(a - bv)B^2 \frac{d^2U_2}{d\tau^2} + U_2(b\omega\kappa - \omega - a\kappa^2) + c_2 U_2^{2n+1} - k_2 U_1 = 0. \tag{59}$$

Based on previous steps, using the balancing procedure between U_l'' and U_l^{2n+1} in Eqs. (58) and (59), we get

$$N + 2 = (2n+1)N \Leftrightarrow 2nN = 2 \Leftrightarrow N = \frac{1}{n}.$$

To obtain an analytic solution, we use the transformation $U_1 = V_1^{\frac{1}{2n}} = V_2^{\frac{1}{2n}} = U_2$ in Eqs. (58) and (59) to find

$$\begin{aligned}(a - bv)B^2\{(1-2n)(V_1')^2 + 2nV_1V_1''\} + 4(b\omega\kappa - \omega - a\kappa^2)n^2V_1^2 \\ + 4c_1n^2V_1^3 - 4k_1n^2V_1^2 = 0,\end{aligned} \tag{60}$$

$$\begin{aligned}(a - bv)B^2\{(1-2n)(V_2')^2 + 2nV_2V_2''\} + 4(b\omega\kappa - \omega - a\kappa^2)n^2V_2^2 \\ + 4c_2n^2V_2^3 - 4k_2n^2V_2^2 = 0.\end{aligned} \tag{61}$$

Balancing the order of V_lV_l'' and V_l^3 in Eqs. (60) and (61), we have $N = 2$. Therefore, Eqs. (60) and (61) have the solutions in the form

$$V_1(\tau) = A_0 + A_1\left(\frac{G'}{G}\right) + A_2\left(\frac{G'}{G}\right)^2, \tag{62}$$

$$V_2(\tau) = B_0 + B_1\left(\frac{G'}{G}\right) + B_2\left(\frac{G'}{G}\right)^2. \tag{63}$$

Substituting Eqs. (62) and (63) along with Eq. (30) in Eqs. (60) and (61) and setting all the coefficients of exponents G'/G to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$\omega = \frac{B^2(bv - a)(\lambda^2 - 4\mu) + 4n^2(a\kappa^2 + k_1)}{4n^2(b\kappa - 1)}, \tag{64}$$

$$A_0 = -\frac{\mu B^2(a - bv)(1+n)}{n^2 c_1}, \tag{65}$$

$$A_1 = -\frac{\lambda B^2(a-bv)(1+n)}{n^2c_1}, \tag{66}$$

$$A_2 = -\frac{B^2(a-bv)(1+n)}{n^2c_1}, \tag{67}$$

$$B_0 = -\frac{\mu B^2(a-bv)(1+n)}{n^2c_1}, \tag{68}$$

$$B_1 = -\frac{\lambda B^2(a-bv)(1+n)}{n^2c_1}, \tag{69}$$

$$B_2 = -\frac{B^2(a-bv)(1+n)}{n^2c_1}, \tag{70}$$

$$k_2 = k_1, \quad c_1 = c_2, \tag{71}$$

where $B,\ \kappa,\ \lambda,\ \mu$ are arbitrary constants.

Now, from Eqs. (20) and (64), we have

$$\omega = \frac{B^2a(b\kappa+1)(\lambda^2-4\mu)+(b\kappa-1)(4n^2a\kappa^2+4n^2k_1)}{4n^2(b\kappa-1)^2+(\lambda^2-4\mu)B^2b^2}. \tag{72}$$

Substituting the solution set (64)–(71) into Eqs. (62) and (63), the solution formulas of Eqs. (60) and (61) can be written as

$$V_1(\tau) = -\frac{B^2(a-bv)(1+n)}{n^2c_1}\left(\mu+\lambda\left(\frac{G'}{G}\right)+\left(\frac{G'}{G}\right)^2\right), \tag{73}$$

$$V_2(\tau) = -\frac{B^2(a-bv)(1+n)}{n^2c_1}\left(\mu+\lambda\left(\frac{G'}{G}\right)+\left(\frac{G'}{G}\right)^2\right). \tag{74}$$

Using the transformation $U_l = V_l^{\frac{1}{2n}}$, we can obtain the following solutions of Eqs. (58) and (59):

$$U_1(\tau) = \left[-\frac{B^2(a-bv)(1+n)}{n^2c_1}\left(\mu+\lambda\left(\frac{G'}{G}\right)+\left(\frac{G'}{G}\right)^2\right)\right]^{\frac{1}{2n}}, \tag{75}$$

$$U_2(\tau) = \left[-\frac{B^2(a-bv)(1+n)}{n^2c_1}\left(\mu+\lambda\left(\frac{G'}{G}\right)+\left(\frac{G'}{G}\right)^2\right)\right]^{\frac{1}{2n}}. \tag{76}$$

Substituting the general solutions of second-order linear ODE into (75) and (76) gives three types of traveling wave solutions.

When $\Delta = \lambda^2 - 4\mu > 0$, we obtain the hyperbolic function traveling wave solutions

$$q(x,t) = \left[\frac{B^2(a-bv)(\lambda^2-4\mu)(1+n)}{4n^2c_1} \times \left(1 - \left(\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}B(x-vt)\right) + C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}B(x-vt)\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}B(x-vt)\right) + C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}B(x-vt)\right)}\right)^2\right)\right]^{\frac{1}{2n}} \times e^{i(-\kappa x+\omega t+\theta)}, \tag{77}$$

$$r(x,t) = \left[\frac{B^2(a-bv)(\lambda^2-4\mu)(1+n)}{4n^2c_1} \times \left(1 - \left(\frac{C_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}B(x-vt)\right) + C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}B(x-vt)\right)}{C_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}B(x-vt)\right) + C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}B(x-vt)\right)}\right)^2\right)\right]^{\frac{1}{2n}} \times e^{i(-\kappa x+\omega t+\theta)}, \tag{78}$$

where C_1 and C_2 are arbitrary constants and ω is given by Eq. (72), while v is given by Eq. (20), and $k_2 = k_1$.

As a special case, assuming $C_1 \neq 0$ and $C_2 = 0$ the traveling wave solution of Eqs. (56) and (57) leads to dark solitons

$$q(x,t) = \left[\frac{B^2(a-bv)(\lambda^2-4\mu)(1+n)}{4n^2c_1}\tanh^2\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}B(x-vt)\right\}\right]^{\frac{1}{2n}} \times e^{i(-\kappa x+\omega t+\theta)}, \tag{79}$$

$$r(x,t) = \left[\frac{B^2(a-bv)(\lambda^2-4\mu)(1+n)}{4n^2c_1}\tanh^2\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}B(x-vt)\right\}\right]^{\frac{1}{2n}} \times e^{i(-\kappa x+\omega t+\theta)}, \tag{80}$$

and assuming $C_1 = 0$ and $C_2 \neq 0$, gives singular solitons:

$$q(x,t) = \left[-\frac{B^2(a-bv)(\lambda^2-4\mu)(1+n)}{4n^2c_1}\coth^2\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}B(x-vt)\right\}\right]^{\frac{1}{2n}} \times e^{i(-\kappa x+\omega t+\theta)}, \tag{81}$$

$$r(x,t) = \left[-\frac{B^2(a-bv)(\lambda^2-4\mu)(1+n)}{4n^2c_1}\coth^2\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}B(x-vt)\right\}\right]^{\frac{1}{2n}} \times e^{i(-\kappa x+\omega t+\theta)}, \tag{82}$$

where ω is given by Eq. (72), while v is given by Eq. (20). These are dark and singular soliton solutions.

When $\Delta = \lambda^2 - 4\mu < 0$, we obtain the trigonometric function traveling wave solution

$$q(x,t) = \left[\frac{B^2(a-bv)(\lambda^2-4\mu)(1+n)}{4n^2c_1} \times \left(1+\left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}B(x-vt)\right)+C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}B(x-vt)\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}B(x-vt)\right)+C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}B(x-vt)\right)}\right)^2\right)\right]^{\frac{1}{2n}} \times e^{i(-\kappa x+\omega t+\theta)}, \tag{83}$$

$$r(x,t) = \left[\frac{B^2(a-bv)(\lambda^2-4\mu)(1+n)}{4n^2c_1} \times \left(1+\left(\frac{-C_1\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}B(x-vt)\right)+C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}B(x-vt)\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}B(x-vt)\right)+C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}B(x-vt)\right)}\right)^2\right)\right]^{\frac{1}{2n}} \times e^{i(-\kappa x+\omega t+\theta)}, \tag{84}$$

where C_1 and C_2 are arbitrary constants and ω is given by Eq. (72), while v is given by Eq. (20).

Again assuming $C_1 \neq 0$ and $C_2 = 0$,

$$q(x,t) = \left[\frac{B^2(a-bv)(\lambda^2-4\mu)(1+n)}{4n^2c_1}\tan^2\left\{\frac{\sqrt{4\mu-\lambda^2}}{2}B(x-vt)\right\}\right]^{\frac{1}{2n}} \times e^{i(-\kappa x+\omega t+\theta)}, \tag{85}$$

$$r(x,t) = \left[\frac{B^2(a-bv)(\lambda^2-4\mu)(1+n)}{4n^2c_1}\tan^2\left\{\frac{\sqrt{4\mu-\lambda^2}}{2}B(x-vt)\right\}\right]^{\frac{1}{2n}} \times e^{i(-\kappa x+\omega t+\theta)}, \tag{86}$$

and on the other hand, $C_1 = 0,\ C_2 \neq 0$ leads to

$$q(x,t) = \left[\frac{B^2(a-bv)(\lambda^2-4\mu)(1+n)}{4n^2c_1}\cot^2\left\{\frac{\sqrt{4\mu-\lambda^2}}{2}B(x-vt)\right\}\right]^{\frac{1}{2n}} \times e^{i(-\kappa x+\omega t+\theta)}, \tag{87}$$

$$r(x,t) = \left[\frac{B^2(a-bv)(\lambda^2-4\mu)(1+n)}{4n^2c_1}\cot^2\left\{\frac{\sqrt{4\mu-\lambda^2}}{2}B(x-vt)\right\}\right]^{\frac{1}{2n}} \times e^{i(-\kappa x+\omega t+\theta)}, \tag{88}$$

where ω is given by Eq. (72) and v is given by Eq. (20), which are singular periodic solutions.

When $\Delta = \lambda^2 - 4\mu = 0$, the plane wave solution is:

$$q(x,t) = \left[-\frac{B^2(a-bv)(1+n)}{n^2c_1}\left\{\frac{C_2}{C_1+C_2B(x-vt)}\right\}^2\right]^{\frac{1}{2n}} e^{i(-\kappa x+\omega t+\theta)}, \tag{89}$$

$$r(x,t) = \left[-\frac{B^2(a-bv)(1+n)}{n^2c_1}\left\{\frac{C_2}{C_1+C_2B(x-vt)}\right\}^2\right]^{\frac{1}{2n}} e^{i(-\kappa x+\omega t+\theta)}, \tag{90}$$

where ω is given by Eq. (72), with v in (20).

3.3. *Parabolic law nonlinearity*

For parabolic law nonlinear media, the governing NLSE is given by[1–6,38]

$$iq_t + aq_{xx} + bq_{xt} + (\xi_1|q|^2 + \eta_1|q|^4)q = k_1 r, \tag{91}$$

$$ir_t + ar_{xx} + br_{xt} + (\xi_2|r|^2 + \eta_2|r|^4)r = k_2 q. \tag{92}$$

The parameters ξ_l and η_l for $l = 1, 2$ represent the coefficients of cubic and quintic nonlinear terms for the two components. In this case, real part equation (23) reduces to

$$(a-bv)B^2\frac{d^2U_1}{d\tau^2}+U_1(b\omega\kappa-\omega-a\kappa^2)+\xi_1U_1^3+\eta_1U_1^5-k_1U_2=0, \tag{93}$$

$$(a-bv)B^2\frac{d^2U_2}{d\tau^2}+U_2(b\omega\kappa-\omega-a\kappa^2)+\xi_2U_2^3+\eta_2U_2^5-k_2U_1=0. \tag{94}$$

Balancing U_l'' with U_l^5 in Eqs. (93) and (94) we have

$$N+2=5N\Leftrightarrow 2=4N\Leftrightarrow N=\frac{1}{2}.$$

We then assume that Eqs. (93) and (94) have the following formal solutions:

$$U_1(\tau)=A_1\left(\frac{G'}{G}\right)^{\frac{1}{2}},\quad A_1\neq 0, \tag{95}$$

$$U_2(\tau)=A_2\left(\frac{G'}{G}\right)^{\frac{1}{2}},\quad A_2\neq 0, \tag{96}$$

where A_1 and A_2 are constants to be determined later and G satisfies Eq. (30). Thus, we obtain

$$\begin{aligned}&(a-bv)B^2\left\{\frac{3}{4}A_1\left(\frac{G'}{G}\right)^{\frac{5}{2}}+A_1\lambda\left(\frac{G'}{G}\right)^{\frac{3}{2}}+\left(\frac{1}{2}A_1\mu+\frac{1}{4}A_1\lambda^2\right)\left(\frac{G'}{G}\right)^{\frac{1}{2}}\right.\\&\qquad\left.-\frac{1}{4}A_1\mu^2\left(\frac{G'}{G}\right)^{-\frac{3}{2}}\right\}\\&\qquad+A_1\left(\frac{G'}{G}\right)^{\frac{1}{2}}(b\omega\kappa-\omega-a\kappa^2)+\xi_1\left\{A_1\left(\frac{G'}{G}\right)^{\frac{1}{2}}\right\}^3\\&\qquad+\eta_1\left\{A_1\left(\frac{G'}{G}\right)^{\frac{1}{2}}\right\}^5-k_1A_2\left(\frac{G'}{G}\right)^{\frac{1}{2}}=0,\end{aligned} \tag{97}$$

$$\begin{aligned}&(a-bv)B^2\left\{\frac{3}{4}A_2\left(\frac{G'}{G}\right)^{\frac{5}{2}}+A_2\lambda\left(\frac{G'}{G}\right)^{\frac{3}{2}}+\left(\frac{1}{2}A_2\mu+\frac{1}{4}A_2\lambda^2\right)\left(\frac{G'}{G}\right)^{\frac{1}{2}}\right.\\&\qquad\left.-\frac{1}{4}A_2\mu^2\left(\frac{G'}{G}\right)^{-\frac{3}{2}}\right\}\\&\qquad+A_2\left(\frac{G'}{G}\right)^{\frac{1}{2}}(b\omega\kappa-\omega-a\kappa^2)+\xi_2\left\{A_2\left(\frac{G'}{G}\right)^{\frac{1}{2}}\right\}^3\\&\qquad+\eta_2\left\{A_2\left(\frac{G'}{G}\right)^{\frac{1}{2}}\right\}^5-k_2A_1\left(\frac{G'}{G}\right)^{\frac{1}{2}}=0.\end{aligned} \tag{98}$$

Then, equating the coefficients of the exponents of G'/G to zero, we obtain a system of nonlinear algebraic equations and by solving it, we get

$$\omega = \frac{3}{16}\frac{\xi_1^2}{\eta_1(b\kappa - 1)} + \frac{\pm k_2\sqrt{\xi_1\xi_2} + a\kappa^2\xi_1}{\xi_1(b\kappa - 1)}, \tag{99}$$

$$v = \frac{3B^2 a\xi_1^2 + 4\eta_1\xi_2^2 A_2^4}{3\xi_1^2 B^2 b}, \tag{100}$$

$$A_1 = \pm\sqrt{\frac{\xi_2}{\xi_1}}A_2, \tag{101}$$

$$k_2 = k_1\frac{\xi_1}{\xi_2}, \tag{102}$$

$$\mu = 0, \tag{103}$$

$$\eta_2 = \frac{\xi_2^2}{\xi_1^2}\eta_1, \tag{104}$$

$$\lambda = \frac{3\xi_1^2}{4\xi_2\eta_1 A_2^2}, \tag{105}$$

where $B,\ \kappa,\ A_2$ are arbitrary constants.

Equating the two expressions for the soliton speed v from (20) and (100) implies

$$B = \pm\frac{2\xi_2 A_2^2}{\xi_1}\sqrt{\frac{\eta_1(1 - b\kappa)}{3(b^2\omega - ab\kappa - a)}}, \tag{106}$$

which immediately prompts the constraint

$$\eta_1(b^2\omega - ab\kappa - a)(1 - b\kappa) > 0. \tag{107}$$

Now, from Eq. (101), we have

$$\frac{A_1}{A_2} = \pm\sqrt{\frac{\xi_2}{\xi_1}} \tag{108}$$

which immediately prompts the constraint

$$\xi_{w1}\xi_2 > 0. \tag{109}$$

By Eq. (104), we get

$$\xi_2^2\eta_1 - \xi_1^2\eta_2 = 0. \tag{110}$$

Thus, we obtain the exact traveling wave solution of Eqs. (91) and (92) as

$$q(x,t) = \left\{-\frac{3\xi_1}{8\eta_1}\left[1 \pm \tanh\left\{\frac{\xi_1}{4}\sqrt{\frac{3(1 - b\kappa)}{\eta_1(b^2\omega - ab\kappa - a)}}(x - vt)\right\}\right]\right\}^{\frac{1}{2}} \times e^{i(-\kappa x + \omega t + \theta)}, \tag{111}$$

$$r(x,t) = \left\{ -\frac{3\xi_1^2}{8\xi_2\eta_1} \left[1 \pm \tanh\left\{ \frac{\xi_1}{4} \sqrt{\frac{3(1-b\kappa)}{\eta_1(b^2\omega - ab\kappa - a)}}(x - vt) \right\} \right] \right\}^{\frac{1}{2}} \times e^{i(-\kappa x + \omega t + \theta)}, \tag{112}$$

which are dark soliton solutions and

$$q(x,t) = \left\{ -\frac{3\xi_1}{8\eta_1} \left[1 \pm \coth\left\{ \frac{\xi_1}{4} \sqrt{\frac{3(1-b\kappa)}{\eta_1(b^2\omega - ab\kappa - a)}}(x - vt) \right\} \right] \right\}^{\frac{1}{2}} \times e^{i(-\kappa x + \omega t + \theta)}, \tag{113}$$

$$r(x,t) = \left\{ -\frac{3\xi_1^2}{8\xi_2\eta_1} \left[1 \pm \coth\left\{ \frac{\xi_1}{4} \sqrt{\frac{3(1-b\kappa)}{\eta_1(b^2\omega - ab\kappa - a)}}(x - vt) \right\} \right] \right\}^{\frac{1}{2}} \times e^{i(-\kappa x + \omega t + \theta)}, \tag{114}$$

which are singular soliton solutions. Here, v is given by Eq. (20), and ω is given by Eq. (99). There are five constraint conditions in order for these analytical solutions to exist, they are given by Eqs. (107)–(110) and $k_2 = (k_1\xi_1)/\xi_2$, respectively.

3.4. *Dual-power law nonlinearity*

For dual-power law nonlinearity, the governing coupled NLSE is[1–6,38]

$$iq_t + aq_{xx} + bq_{xt} + (\xi_1|q|^{2n} + \eta_1|q|^{4n})q = k_1 r, \tag{115}$$

$$ir_t + ar_{xx} + br_{xt} + (\xi_2|r|^{2n} + \eta_2|r|^{4n})r = k_2 q. \tag{116}$$

The special case, for $n = 1$, is parabolic law nonlinearity, discussed in the previous sub-section. In this case, real part equation (23) reduces to

$$(a - bv)B^2\frac{d^2U_1}{d\tau^2} + U_1(b\omega\kappa - \omega - a\kappa^2) + \xi_1 U_1^{2n+1} + \eta_1 U_1^{4n+1} - k_1 U_2 = 0, \tag{117}$$

$$(a - bv)B^2\frac{d^2U_2}{d\tau^2} + U_2(b\omega\kappa - \omega - a\kappa^2) + \xi_2 U_2^{2n+1} + \eta_2 U_2^{4n+1} - k_2 U_1 = 0. \tag{118}$$

Balancing U_l'' with U_l^{4n+1} in Eqs. (117) and (118) we have

$$N + 2 = (4n+1)N \Leftrightarrow 2 = 4nN \Leftrightarrow N = \frac{1}{2n}.$$

We then assume that Eqs. (117) and (118) have the following formal solutions:

$$U_1(\tau) = A_1\left(\frac{G'}{G}\right)^{\frac{1}{2n}}, \quad A_1 \neq 0, \tag{119}$$

$$U_2(\tau) = A_2\left(\frac{G'}{G}\right)^{\frac{1}{2n}}, \quad A_2 \neq 0, \tag{120}$$

where A_1 and A_2 are constants to be determined later and G satisfies Eq. (30). Thus, we obtain

$$\begin{aligned}
&(a-bv)B^2\left\{\left(\frac{1}{4n^2}+\frac{1}{2n}\right)A_1\left(\frac{G'}{G}\right)^{\frac{1}{2n}+2}+\left(\frac{1}{2n^2}+\frac{1}{2n}\right)A_1\lambda\left(\frac{G'}{G}\right)^{\frac{1}{2n}+1}\right.\\
&\quad+\left(\frac{1}{2n^2}A_1\mu+\frac{1}{4n^2}A_1\lambda^2\right)\left(\frac{G'}{G}\right)^{\frac{1}{2n}}+\left(\frac{1}{2n^2}-\frac{1}{2n}\right)A_1\mu\lambda\left(\frac{G'}{G}\right)^{\frac{1}{2n}-1}\\
&\quad\left.+\left(\frac{1}{4n^2}-\frac{1}{2n}\right)A_1\mu^2\left(\frac{G'}{G}\right)^{\frac{1}{2n}-2}\right\}\\
&\quad+A_1\left(\frac{G'}{G}\right)^{\frac{1}{2n}}(b\omega\kappa-\omega-a\kappa^2)+\xi_1\left\{A_1\left(\frac{G'}{G}\right)^{\frac{1}{2n}}\right\}^{2n+1}\\
&\quad+\eta_1\left\{A_1\left(\frac{G'}{G}\right)^{\frac{1}{2n}}\right\}^{4n+1}-k_1A_2\left(\frac{G'}{G}\right)^{\frac{1}{2n}}=0, \qquad (121)
\end{aligned}$$

$$\begin{aligned}
&(a-bv)B^2\left\{\left(\frac{1}{4n^2}+\frac{1}{2n}\right)A_2\left(\frac{G'}{G}\right)^{\frac{1}{2n}+2}+\left(\frac{1}{2n^2}+\frac{1}{2n}\right)A_2\lambda\left(\frac{G'}{G}\right)^{\frac{1}{2n}+1}\right.\\
&\quad+\left(\frac{1}{2n^2}A_2\mu+\frac{1}{4n^2}A_2\lambda^2\right)\left(\frac{G'}{G}\right)^{\frac{1}{2n}}+\left(\frac{1}{2n^2}-\frac{1}{2n}\right)A_2\mu\lambda\left(\frac{G'}{G}\right)^{\frac{1}{2n}-1}\\
&\quad\left.+\left(\frac{1}{4n^2}-\frac{1}{2n}\right)A_2\mu^2\left(\frac{G'}{G}\right)^{\frac{1}{2n}-2}\right\}\\
&\quad+A_2\left(\frac{G'}{G}\right)^{\frac{1}{2n}}(b\omega\kappa-\omega-a\kappa^2)+\xi_2\left\{A_2\left(\frac{G'}{G}\right)^{\frac{1}{2n}}\right\}^{2n+1}\\
&\quad+\eta_2\left\{A_2\left(\frac{G'}{G}\right)^{\frac{1}{2n}}\right\}^{4n+1}-k_2A_1\left(\frac{G'}{G}\right)^{\frac{1}{2n}}=0. \qquad (122)
\end{aligned}$$

Then, equating the coefficient of each power of G'/G to zero, we obtain a system of nonlinear algebraic equations and by solving it, reduces to

$$\omega=\frac{1+2n}{4(n+1)^2}\frac{\xi_1^2}{\eta_1(b\kappa-1)}+\frac{\pm k_2\sqrt{\xi_1\xi_2}+a\kappa^2\xi_1}{\xi_1(b\kappa-1)}, \qquad (123)$$

$$v=\frac{(1+2n)B^2a\xi_1^2+4n^2\eta_1\xi_2^2A_2^{4n}}{(1+2n)\xi_1^2B^2b}, \qquad (124)$$

$$A_1=\left(\frac{\xi_2}{\xi_1}\right)^{\frac{1}{2n}}A_2, \qquad (125)$$

$$k_2 = k_1 \frac{\xi_1}{\xi_2}, \tag{126}$$

$$\mu = 0, \tag{127}$$

$$\eta_2 = \frac{\xi_2^2}{\xi_1^2}\eta_1, \tag{128}$$

$$\lambda = \frac{(1+2n)\xi_1^2}{2(1+n)\xi_2\eta_1 A_2^{2n}}, \tag{129}$$

where $B,\ \kappa,\ A_2$ are arbitrary constants.

Equating the two expressions for the soliton speed v from (20) and (124) implies

$$B = \pm\frac{2n\xi_2 A_2^{2n}}{\xi_1}\sqrt{\frac{\eta_1(1-b\kappa)}{(1+2n)(b^2\omega - ab\kappa - a)}}, \tag{130}$$

which immediately prompts the constraint

$$\eta_1(b^2\omega - ab\kappa - a)(1-b\kappa) > 0. \tag{131}$$

Now, from Eq. (125), we have

$$\frac{A_1}{A_2} = \left(\frac{\xi_2}{\xi_1}\right)^{\frac{1}{2n}} \tag{132}$$

which immediately prompts the constraint

$$\xi_1\xi_2 > 0. \tag{133}$$

By Eq. (128), we get

$$\xi_2^2\eta_1 - \xi_1^2\eta_2 = 0. \tag{134}$$

Thus, we obtain the exact traveling wave solution of Eqs. (115) and (116) as

$$q(x,t) = \left\{-\frac{(1+2n)\xi_1}{4(1+n)\eta_1}\left[1 \pm \tanh\left\{\frac{n\xi_1}{2(1+n)}\sqrt{\frac{(1+2n)(1-b\kappa)}{\eta_1(b^2\omega - ab\kappa - a)}}(x-vt)\right\}\right]\right\}^{\frac{1}{2n}} \times e^{i(-\kappa x+\omega t+\theta)}, \tag{135}$$

$$r(x,t) = \left\{-\frac{(1+2n)\xi_1^2}{4(1+n)\xi_2\eta_1}\left[1 \pm \tanh\left\{\frac{n\xi_1}{2(1+n)}\sqrt{\frac{(1+2n)(1-b\kappa)}{\eta_1(b^2\omega - ab\kappa - a)}}(x-vt)\right\}\right]\right\}^{\frac{1}{2n}} \times e^{i(-\kappa x+\omega t+\theta)}, \tag{136}$$

which are dark 1-soliton solutions. Then the singular 1-soliton solutions are:

$$q(x,t) = \left\{-\frac{(1+2n)\xi_1}{4(1+n)\eta_1}\left[1 \pm \coth\left\{\frac{n\xi_1}{2(1+n)}\sqrt{\frac{(1+2n)(1-b\kappa)}{\eta_1(b^2\omega - ab\kappa - a)}}(x-vt)\right\}\right]\right\}^{\frac{1}{2n}} \times e^{i(-\kappa x+\omega t+\theta)}, \tag{137}$$

$$r(x,t) = \left\{ -\frac{(1+2n)\xi_1^2}{4(1+n)\xi_2\eta_1} \left[1 \pm \coth\left\{ \frac{n\xi_1}{2(1+n)} \sqrt{\frac{(1+2n)(1-b\kappa)}{\eta_1(b^2\omega - ab\kappa - a)}} (x - vt) \right\} \right] \right\}^{\frac{1}{2n}} \times e^{i(-\kappa x + \omega t + \theta)}, \tag{138}$$

where v is given by Eq. (20), and ω is given by Eq. (123). There are five constraint conditions in order for these analytical solutions to exist, they are given by Eqs. (131)–(134) and $k_2 = (k_1\xi_1)/\xi_2$, respectively.

4. Multiple-Core Couplers (Coupling with Nearest Neighbors)

The governing equation for twin-core couplers is given by[1−6,38]

$$iq_t^{(l)} + a_l q_{xx}^{(l)} + b_l q_{xt}^{(l)} + c_l F(|q^{(l)}|^2) q^{(l)} = k_l[q^{(l-1)} - 2q^{(l)} + q^{(l+1)}], \tag{139}$$

where $1 \le l \le N$. Equation (139) represents the general model for optical couplers where coupling with nearest neighbors is considered. Here k_l are, as before, the coupling coefficients. In order to address this model for the four forms of nonlinear media, the initial hypothesis is taken to be

$$q^{(l)}(x,t) = P_l(x,t) e^{i\phi(x,t)}, \tag{140}$$

where the amplitude component of soliton is $P_l(x,t)$ while the amplitude component carries the same definition as in (9) or (10). After substituting this hypothesis (140) into (139), the resulting expression is split into real and imaginary components. The imaginary part gives the speed of the soliton as

$$v = \frac{b_l\omega - 2a_l\kappa}{1 - b_l\kappa}. \tag{141}$$

The speed of the soliton stays the same irrespective of the type of nonlinearity and type of solitons that is going to be addressed. Next, the real part implies

$$a_l \frac{\partial^2 P_l}{\partial x^2} + b_l \frac{\partial^2 P_l}{\partial x \partial t} + P_l(b_l\omega\kappa - \omega - a_l\kappa^2) + c_l F(P_l^2) P_l - k_l[P_{l-1} - 2P_l + P_{l+1}] = 0. \tag{142}$$

Under the traveling wave transformation

$$P_l(x,t) = U_l(\tau), \quad \tau = B(x - vt) \tag{143}$$

we have

$$(a_l - b_l v) B^2 \frac{d^2 U_l}{d\tau^2} + U_l(b_l\omega\kappa - \omega - a_l\kappa^2) + c_l F(U_l^2) U_l - k_l[U_{l-1} - 2U_l + U_{l+1}] = 0. \tag{144}$$

4.1. *Kerr law nonlinearity*

For Kerr law, the coupled NLSE modifies to

$$iq_t^{(l)} + a_l q_{xx}^{(l)} + b_l q_{xt}^{(l)} + c_l |q^{(l)}|^2 q^{(l)} = k_l[q^{(l-1)} - 2q^{(l)} + q^{(l+1)}]. \tag{145}$$

For hypothesis given by (140) and (143), Eq. (139) reduces to

$$(a_l - b_l v)B^2 \frac{d^2 U_l}{d\tau^2} + U_l(b_l\omega\kappa - \omega - a_l\kappa^2) + c_l U_l^3 - k_l[U_{l-1} - 2U_l + U_{l+1}] = 0. \tag{146}$$

We then assume that Eq. (146) has the following formal solution:

$$U_l(\tau) = A_l\left(\frac{G'}{G}\right), \quad A_l \neq 0, \tag{147}$$

where A_l are constants to be determined later and G satisfies Eq. (30). Thus, we obtain

$$(a_l - b_l v)B^2\left\{2A_l\left(\frac{G'}{G}\right)^3 + 3A_l\lambda\left(\frac{G'}{G}\right)^2 + (2A_l\mu + A_l\lambda^2)\left(\frac{G'}{G}\right) + \lambda\mu A_l\right\} + A_l\left(\frac{G'}{G}\right)(b_l\omega\kappa - \omega - a_l\kappa^2) + c_l A_l^3\left(\frac{G'}{G}\right)^3 - k_l(A_{l-1} - 2A_l + A_{l+1})\left(\frac{G'}{G}\right) = 0. \tag{148}$$

Then, equating the coefficient of each power of G'/G to zero, we obtain a system of nonlinear algebraic equations and by solving it, we get

$$v = \frac{2B^2 a_l + c_l A_l^2}{2B^2 b_l}, \tag{149}$$

$$\lambda = 0, \tag{150}$$

$$\omega = \frac{\mu c_l A_l^3 + A_l a_l \kappa^2 + k_l(A_{l-1} - 2A_l + A_{l+1})}{A_l(b_l\kappa - 1)}, \tag{151}$$

where $B,\ \kappa,\ A_l,\ k_l,\ \mu$ are arbitrary constants.

Next, equating the two values of the speed v from the imaginary part equation (141) and real part equation (149) gives the free parameter

$$B = \pm\frac{A_l(b_l\kappa - 1)\sqrt{-A_l c_l}}{\sqrt{\begin{array}{c}2(b_l^2\mu c_l A_l^3 + b_l^2 A_l a_l\kappa^2 + b_l^2 k_l(A_{l-1} - 2A_l + A_{l+1}) \\ - A_l a_l b_l\kappa(b_l\kappa - 1) - A_l a_l(b_l\kappa - 1))\end{array}}}, \tag{152}$$

which again poses the constraint

$$A_l c_l(b_l^2\mu c_l A_l^3 + b_l^2 A_l a_l\kappa^2 + b_l^2 k_l(A_{l-1} - 2A_l + A_{l+1}) - A_l a_l b_l\kappa(b_l\kappa - 1) - A_l a_l(b_l\kappa - 1)) < 0. \tag{153}$$

Substituting the general solutions of the second-order LODE into the formula (147), we have three types of traveling wave solutions of Eq. (145) as follows:

When $\mu < 0$, we obtain the hyperbolic function traveling wave solution

$$q^{(l)}(x,t) = A_l\sqrt{-\mu}\left(\frac{C_1\sinh(\sqrt{-\mu}(x-vt)) + C_2\cosh(\sqrt{-\mu}(x-vt))}{C_1\cosh(\sqrt{-\mu}(x-vt)) + C_2\sinh(\sqrt{-\mu}(x-vt))}\right) \times e^{i(-\kappa x+\omega t+\theta)}, \tag{154}$$

where C_1 and C_2 are arbitrary constants and parameter B is given by (152), while the speed of the soliton is (141) or (149) and finally the wave number of the soliton is dictated by (151). These solitons will exist provided the constraint condition given by (153) holds. On the other hand, assuming $C_1 \neq 0$ and $C_2 = 0$, the dark 1-soliton solution that falls out is:

$$q^{(l)}(x,t) = A_l\sqrt{-\mu}\tanh\{\sqrt{-\mu}(x-vt)\}e^{i(-\kappa x+\omega t+\theta)}, \tag{155}$$

and again assuming $C_1 = 0$ and $C_2 \neq 0$, then we obtain singular 1-soliton solution for Eq. (145) as

$$q^{(l)}(x,t) = A_l\sqrt{-\mu}\coth\{\sqrt{-\mu}(x-vt)\}e^{i(-\kappa x+\omega t+\theta)}, \tag{156}$$

where parameter B is given by (152), while the speed of the soliton is (141) or (149) and finally the wave number of the soliton is dictated by (151). These solitons will exist if the constraint condition given by (153) holds.

When $\mu > 0$, traveling wave solution is:

$$q^{(l)}(x,t) = A_l\sqrt{\mu}\left(\frac{-C_1\sin(\sqrt{\mu}(x-vt)) + C_2\cos(\sqrt{\mu}(x-vt))}{C_1\cos(\sqrt{\mu}(x-vt)) + C_2\sin(\sqrt{\mu}(x-vt))}\right) \times e^{i(-\kappa x+\omega t+\theta)}, \tag{157}$$

where C_1 and C_2 are arbitrary constants and parameter B is given by (152), while the speed of the soliton is (141) or (149) and finally the wave number of the soliton is dictated by (151). These solitons will exist as long as the constraint condition given by (153) stays.

Also, with the assumption $C_1 \neq 0$ and $C_2 = 0$, the singular periodic solutions are

$$q^{(l)}(x,t) = -A_l\sqrt{\mu}\tan\{\sqrt{\mu}(x-vt)\}e^{i(-\kappa x+\omega t+\theta)}, \tag{158}$$

and when $C_1 = 0,\ C_2 \neq 0$

$$q^{(l)}(x,t) = A_l\sqrt{\mu}\cot\{\sqrt{\mu}(x-vt)\}e^{i(-\kappa x+\omega t+\theta)}, \tag{159}$$

where parameter B is given by (152), while the speed of the soliton is (141) or (149) and finally the wave number is given by (151). These are singular periodic solutions that exist whenever the constraint conditions given by (153) hold.

For $\mu = 0$, we obtain plane wave solution

$$q^{(l)}(x,t) = \frac{A_l C_2}{C_1 + C_2 B(x-vt)}e^{i(-\kappa x+\omega t+\theta)}, \tag{160}$$

where parameter B is given by (152), while the speed of the soliton is (141) or (149) and finally the wave number of the soliton is dictated by (151). These solitons will exist provided the constraint condition given by (153) holds.

4.2. *Power law nonlinearity*

For power law, the coupled NLSE modifies to

$$iq_t^{(l)} + a_l q_{xx}^{(l)} + b_l q_{xt}^{(l)} + c_l |q^{(l)}|^{2n} q^{(l)} = k_l [q^{(l-1)} - 2q^{(l)} + q^{(l+1)}]. \tag{161}$$

In this case, Eq. (124) gives

$$(a_l - b_l v) B^2 \frac{d^2 U_l}{d\tau^2} + U_l (b_l \omega \kappa - \omega - a_l \kappa^2) + c_l U_l^{2n+1} - k_l [U_{l-1} - 2U_l + U_{l+1}] = 0. \tag{162}$$

According to the previous steps, using the balancing procedure between U_l'' and U_l^{2n+1} in Eq. (162), we get

$$N + 2 = (2n+1)N \Leftrightarrow 2nN = 2 \Leftrightarrow N = \frac{1}{n}.$$

We then assume that Eq. (162) has the following formal solution:

$$U_l(\tau) = A_l \left(\frac{G'}{G}\right)^{\frac{1}{n}}, \quad A_l \neq 0, \tag{163}$$

where A_l are constants to be determined later and G satisfies Eq. (30). Thus, we obtain

$$\begin{aligned} &(a_l - b_l v) B^2 \left\{ \left(\frac{1}{n^2} + \frac{1}{n}\right) A_l \left(\frac{G'}{G}\right)^{\frac{1}{n}+2} + \left(\frac{2}{n^2} + \frac{1}{n}\right) A_l \lambda \left(\frac{G'}{G}\right)^{\frac{1}{n}+1} \right. \\ &\quad + \left(\frac{2}{n^2} A_l \mu + \frac{1}{n^2} A_l \lambda^2\right) \left(\frac{G'}{G}\right)^{\frac{1}{n}} + \left(\frac{2}{n^2} - \frac{1}{n}\right) A_l \mu \lambda \left(\frac{G'}{G}\right)^{\frac{1}{n}-1} \\ &\quad \left. + \left(\frac{1}{n^2} - \frac{1}{n}\right) A_l \mu^2 \left(\frac{G'}{G}\right)^{\frac{1}{n}-2} \right\} \\ &\quad + A_l \left(\frac{G'}{G}\right)^{\frac{1}{n}} (b_l \omega \kappa - \omega - a_l \kappa^2) + c_l A_l^{2n+1} \left(\frac{G'}{G}\right)^{\frac{1}{n}+2} \\ &\quad - k_l (A_{l-1} - 2A_l + A_{l+1}) \left(\frac{G'}{G}\right)^{\frac{1}{n}} = 0. \end{aligned} \tag{164}$$

Then, equating the coefficient of each power of G'/G to zero, we obtain a system of nonlinear algebraic equations and by solving it, we get

$$v = \frac{(1+n) B^2 a_l + c_l n^2 A_l^{2n}}{(1+n) B^2 b_l}, \tag{165}$$

$$\lambda = \mu = 0, \tag{166}$$

$$\omega = \frac{A_l a_l \kappa^2 + k_l(A_{l-1} - 2A_l + A_{l+1})}{A_l(b_l\kappa - 1)}, \tag{167}$$

where $B,\ \kappa,\ A_l,\ k_l$ are arbitrary constants. Next, equating the two values of the speed v from the imaginary part equation (141) and real part equation (165) gives the free parameter

$$B = \pm \frac{nA_l^n(b_l\kappa - 1)\sqrt{-A_l c_l}}{\sqrt{\begin{array}{c}(1+n)(b_l^2 A_l a_l \kappa^2 + b_l^2 k_l(A_{l-1} - 2A_l + A_{l+1}) \\ - A_l a_l b_l \kappa(b_l\kappa - 1) - A_l a_l(b_l\kappa - 1))\end{array}}}, \tag{168}$$

which again poses the constraint

$$\begin{aligned} &A_l c_l(b_l^2 A_l a_l \kappa^2 + b_l^2 k_l(A_{l-1} - 2A_l + A_{l+1}) \\ &\quad - A_l a_l b_l \kappa(b_l\kappa - 1) - A_l a_l(b_l\kappa - 1)) < 0. \end{aligned} \tag{169}$$

Since $\lambda = \mu = 0$, we obtain plane wave solution

$$q^{(l)}(x,t) = A_l \left\{ \frac{C_2}{C_1 + C_2 B(x - vt)} \right\}^{\frac{1}{n}} e^{i(-\kappa x + \omega t + \theta)}, \tag{170}$$

where C_1 and C_2 are arbitrary constants and parameter B is given by (168), while the speed of the soliton is (141) or (165) and finally the wave number of the soliton is dictated by (167). These solitons will exist with the constraint condition given by (169).

On the other hand, to obtain an analytic solution, we use the transformations $U_l = V_l^{\frac{1}{2n}},\ U_{l-1} = U_{l+1} = -U_l$ in Eq. (162) to find

$$\begin{aligned} &(a_l - b_l v)B^2\{(1-2n)(V_l')^2 + 2nV_l V_l''\} \\ &\quad + 4(b_l\omega\kappa - \omega - a_l\kappa^2)n^2 V_l^2 + 4c_l n^2 V_l^3 + 16k_l n^2 V_l^2 = 0. \end{aligned} \tag{171}$$

Balancing the order of $V_l V_l''$ and V_l^3 in Eq. (171), we have $N = 2$. Therefore, Eq. (171) has the solutions in the form

$$V_l(\tau) = A_{l0} + A_{l1}\left(\frac{G'}{G}\right) + A_{l2}\left(\frac{G'}{G}\right)^2. \tag{172}$$

Substituting Eq. (172) along with Eq. (30) in Eq. (171) and setting all the coefficients of powers G'/G to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$\omega = \frac{B^2(b_l v - a_l)(\lambda^2 - 4\mu) + 4n^2(a_l\kappa^2 - 4k_l)}{4n^2(b_l\kappa - 1)}, \tag{173}$$

$$A_{l0} = -\frac{\mu B^2(a_l - b_l v)(1+n)}{n^2 c_l}, \tag{174}$$

$$A_{l1} = -\frac{\lambda B^2(a_l - b_l v)(1+n)}{n^2 c_l}, \tag{175}$$

$$A_{l2} = -\frac{B^2(a_l - b_l v)(1+n)}{n^2 c_l}, \tag{176}$$

where $B,\ \kappa,\ \lambda,\ \mu$ are arbitrary constants.

Now, from Eqs. (141) and (173), we have

$$\omega = \frac{B^2 a_l(b_l\kappa + 1)(\lambda^2 - 4\mu) + (b_l\kappa - 1)(4n^2 a_l\kappa^2 - 16n^2 k_l)}{4n^2(b_l\kappa - 1)^2 + (\lambda^2 - 4\mu)B^2 b_l^2}. \tag{177}$$

Substituting the general solutions of second-order LODE into (172) gives three types of traveling wave solutions.

When $\Delta = \lambda^2 - 4\mu > 0$, we obtain the hyperbolic function traveling wave solution

$$q^{(l)}(x,t) = \left[\frac{B^2(a_l - b_l v)(\lambda^2 - 4\mu)(1+n)}{4n^2 c_l} \times \left(1 - \left(\frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}B(x-vt)\right) + C_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}B(x-vt)\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}B(x-vt)\right) + C_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}B(x-vt)\right)}\right)^2\right)\right]^{\frac{1}{2n}} \times e^{i(-\kappa x + \omega t + \theta)}, \tag{178}$$

where C_1 and C_2 are arbitrary constants and ω is given by Eq. (177), while v is given by Eq. (141).

On the other hand, assuming $C_1 \neq 0$ and $C_2 = 0$ the dark soliton solution is:

$$q^{(l)}(x,t) = \left[\frac{B^2(a_l - b_l v)(\lambda^2 - 4\mu)(1+n)}{4n^2 c_l}\tanh^2\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}B(x-vt)\right\}\right]^{\frac{1}{2n}} \times e^{i(-\kappa x + \omega t + \theta)}, \tag{179}$$

and assuming $C_1 = 0$ and $C_2 \neq 0$, the singular 1-soliton solution is

$$q^{(l)}(x,t) = \left[-\frac{B^2(a_l - b_l v)(\lambda^2 - 4\mu)(1+n)}{4n^2 c_l}\coth^2\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}B(x-vt)\right\}\right]^{\frac{1}{2n}} \times e^{i(-\kappa x + \omega t + \theta)}, \tag{180}$$

where ω is given by Eq. (177), while v is given by Eq. (141).

When $\Delta = \lambda^2 - 4\mu < 0$, we obtain trigonometric function traveling wave solution

$$q^{(l)}(x,t) = \left[\frac{B^2(a_l - b_l v)(\lambda^2 - 4\mu)(1+n)}{4n^2 c_l} \times \left(1 + \left(\frac{-C_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}B(x-vt)\right) + C_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}B(x-vt)\right)}{C_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}B(x-vt)\right) + C_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}B(x-vt)\right)}\right)^2\right)\right]^{\frac{1}{2n}} \times e^{i(-\kappa x + \omega t + \theta)}, \tag{181}$$

where C_1 and C_2 are arbitrary constants and ω is given by Eq. (177), while v is given by Eq. (141).

Also, upon assuming $C_1 \neq 0$ and $C_2 = 0$, the following singular periodic solutions are revealed:

$$q^{(l)}(x,t) = \left[\frac{B^2(a_l - b_l v)(\lambda^2 - 4\mu)(1+n)}{4n^2 c_l}\tan^2\left\{\frac{\sqrt{4\mu-\lambda^2}}{2}B(x-vt)\right\}\right]^{\frac{1}{2n}} \times e^{i(-\kappa x + \omega t + \theta)}, \tag{182}$$

and when $C_1 = 0,\ C_2 \neq 0$

$$q^{(l)}(x,t) = \left[\frac{B^2(a_l - b_l v)(\lambda^2 - 4\mu)(1+n)}{4n^2 c_l}\cot^2\left\{\frac{\sqrt{4\mu-\lambda^2}}{2}B(x-vt)\right\}\right]^{\frac{1}{2n}} \times e^{i(-\kappa x + \omega t + \theta)}, \tag{183}$$

where ω is given by Eq. (177), while v is given by Eq. (141).

Finally, for $\Delta = \lambda^2 - 4\mu = 0$, we obtain plane wave solution:

$$q^{(l)}(x,t) = \left[-\frac{B^2(a_l - b_l v)(1+n)}{n^2 c_l}\left\{\frac{C_2}{C_1 + C_2 B(x-vt)}\right\}^2\right]^{\frac{1}{2n}} \times e^{i(-\kappa x + \omega t + \theta)}, \tag{184}$$

where ω is given by Eq. (177), while v is given by Eq. (141).

4.3. *Parabolic law nonlinearity*

In this case, the governing equation reduces to[1−6,38]

$$iq_t^{(l)} + a_l q_{xx}^{(l)} + b_l q_{xt}^{(l)} + (\xi_l |q^{(l)}|^2 + \eta_l |q^{(l)}|^4) q^{(l)} = k_l[q^{(l-1)} - 2q^{(l)} + q^{(l+1)}], \tag{185}$$

where $1 \le l \le N$. The real part equation therefore is

$$(a_l - b_l v)B^2 \frac{d^2 U_l}{d\tau^2} + U_l(b_l \omega\kappa - \omega - a_l\kappa^2) \\ + \xi_l U_l^3 + \eta_l U_l^5 - k_l(U_{l-1} - 2U_l + U_{l+1}) = 0. \tag{186}$$

We then assume that Eq. (186) has the following formal solutions:

$$U_l(\tau) = A_l\left(\frac{G'}{G}\right)^{\frac{1}{2}}, \quad A_l \neq 0, \tag{187}$$

where A_l are constants to be determined later and G satisfies Eq. (30). Thus, we obtain

$$(a_l - b_l v)B^2\left\{\frac{3}{4}A_l\left(\frac{G'}{G}\right)^{\frac{5}{2}} + A_l\lambda\left(\frac{G'}{G}\right)^{\frac{3}{2}} \right. \\ \left. + \left(\frac{1}{2}A_l\mu + \frac{1}{4}A_l\lambda^2\right)\left(\frac{G'}{G}\right)^{\frac{1}{2}} - \frac{1}{4}A_l\mu^2\left(\frac{G'}{G}\right)^{-\frac{3}{2}}\right\} \\ + A_l\left(\frac{G'}{G}\right)^{\frac{1}{2}}(b_l\omega\kappa - \omega - a_l\kappa^2) + \xi_l\left\{A_l\left(\frac{G'}{G}\right)^{\frac{1}{2}}\right\}^3 + \eta_l\left\{A_l\left(\frac{G'}{G}\right)^{\frac{1}{2}}\right\}^5 \\ - k_l(A_{l-1} - 2A_l + A_{l+1})\left(\frac{G'}{G}\right)^{\frac{1}{2}} = 0. \tag{188}$$

Then, equating the coefficient of each power of G'/G to zero, we obtain a system of nonlinear algebraic equations and by solving it, we get

$$v = \frac{4\eta_l A_l^4 + 3B^2 a_l}{3B^2 b_l}, \tag{189}$$

$$\lambda = \frac{3\xi_l}{4A_l^2\eta_l}, \tag{190}$$

$$\mu = 0, \tag{191}$$

$$\omega = \frac{3\xi_l^2 A_l + 16A_l a_l\kappa^2\eta_l + 16k_l\eta_l(A_{l-1} - 2A_l + A_{l+1})}{16A_l\eta_l(b_l\kappa - 1)}, \tag{192}$$

where $B,\ \kappa,\ A_l,\ k_l$ are arbitrary constants.

Now, equating the two values of the speed v from the imaginary part equation (141) and real part equation (189) gives the free parameter

$$B = \pm\frac{2A_l^2\sqrt{\eta_l(1 - b_l\kappa)}}{\sqrt{3(b_l^2\omega - a_l b_l\kappa - a_l)}} \tag{193}$$

which immediately prompts the constraint

$$\eta_l(1 - b_l\kappa)(b_l^2\omega - a_l b_l\kappa - a_l) > 0. \tag{194}$$

Thus, we obtain the exact traveling wave solution of Eq. (185) as

$$q^{(l)}(x,t) = \left\{-\frac{3\xi_l}{8\eta_l}\left[1 \pm \tanh\left\{\frac{\xi_l}{4}\sqrt{\frac{3(1-b_l\kappa)}{\eta_l(b_l^2\omega - a_lb_l\kappa - a_l)}}(x-vt)\right\}\right]\right\}^{\frac{1}{2}} \times e^{i(-\kappa x+\omega t+\theta)}, \tag{195}$$

which is a dark 1-soliton solution and

$$q^{(l)}(x,t) = \left\{-\frac{3\xi_l}{8\eta_l}\left[1 \pm \coth\left\{\frac{\xi_l}{4}\sqrt{\frac{3(1-b_l\kappa)}{\eta_l(b_l^2\omega - a_lb_l\kappa - a_l)}}(x-vt)\right\}\right]\right\}^{\frac{1}{2}} \times e^{i(-\kappa x+\omega t+\theta)}, \tag{196}$$

which is a singular 1-soliton solution. Here, the parameter B is given by (193), and speed of the soliton is (141) or (189) and the wave number of the soliton is dictated by (192). These solitons will exist provided the constraint condition given by (194) holds.

4.4. *Dual-power law nonlinearity*

For dual-power law nonlinearity, the governing equation is[1−6,38]

$$iq_t^{(l)} + a_lq_{xx}^{(l)} + b_lq_{xt}^{(l)} + (\xi_l|q^{(l)}|^{2n} + \eta_l|q^{(l)}|^{4n})q^{(l)} = k_l[q^{(l-1)} - 2q^{(l)} + q^{(l+1)}], \tag{197}$$

where $1 \leq l \leq N$. The real part equation therefore is

$$(a_l - b_lv)B^2\frac{d^2U_l}{d\tau^2} + U_l(b_l\omega\kappa - \omega - a_l\kappa^2) + \xi_lU_l^{2n+1} + \eta_lU_l^{4n+1} - k_l(U_{l-1} - 2U_l + U_{l+1}) = 0. \tag{198}$$

We then assume that Eq. (197) has the following formal solutions:

$$U_l(\tau) = A_l\left(\frac{G'}{G}\right)^{\frac{1}{2n}}, \quad A_l \neq 0, \tag{199}$$

where A_l are constants to be determined later and G satisfies Eq. (30). Thus, we obtain

$$(a_l - b_lv)B^2\left\{\left(\frac{1}{4n^2}+\frac{1}{2n}\right)A_l\left(\frac{G'}{G}\right)^{\frac{1}{2n}+2} + \left(\frac{1}{2n^2}+\frac{1}{2n}\right)A_1\lambda\left(\frac{G'}{G}\right)^{\frac{1}{2n}+1} + \left(\frac{1}{2n^2}A_l\mu + \frac{1}{4n^2}A_l\lambda^2\right)\left(\frac{G'}{G}\right)^{\frac{1}{2n}} + \left(\frac{1}{2n^2}-\frac{1}{2n}\right)A_l\mu\lambda\left(\frac{G'}{G}\right)^{\frac{1}{2n}-1} + \left(\frac{1}{4n^2}-\frac{1}{2n}\right)A_l\mu^2\left(\frac{G'}{G}\right)^{\frac{1}{2n}-2}\right\}$$

$$+ A_l\left(\frac{G'}{G}\right)^{\frac{1}{2n}}\left(b_l\omega\kappa - \omega - a_l\kappa^2\right) + \xi_l\left\{A_l\left(\frac{G'}{G}\right)^{\frac{1}{2n}}\right\}^{2n+1}$$

$$+ \eta_l\left\{A_l\left(\frac{G'}{G}\right)^{\frac{1}{2n}}\right\}^{4n+1} - k_l(A_{l-1} - 2A_l + A_{l+1})\left(\frac{G'}{G}\right)^{\frac{1}{2n}} = 0. \tag{200}$$

Then, equating the coefficient of each power of G'/G to zero, we obtain a system of nonlinear algebraic equations and by solving it, we get

$$v = \frac{4n^2\eta_l A_l^{4n} + (1+2n)B^2 a_l}{(1+2n)B^2 b_l}, \tag{201}$$

$$\lambda = \frac{(1+2n)\xi_l}{2(1+n)A_l^{2n}\eta_l}, \tag{202}$$

$$\mu = 0, \tag{203}$$

$$\omega = \frac{(1+2n)\xi_l^2 A_l + 4(1+n)^2 A_l a_l \kappa^2 \eta_l + 4(1+n)^2 k_l \eta_l (A_{l-1} - 2A_l + A_{l+1})}{4(1+n)^2 A_l \eta_l (b_l\kappa - 1)}, \tag{204}$$

where $B,\ \kappa,\ A_l,\ k_l$ are arbitrary constants.

Next, equating the two values of the speed v from the imaginary part equation (141) and real part equation (201) gives the free parameter

$$B = \pm\frac{2nA_l^{2n}\sqrt{\eta_l(1-b_l\kappa)}}{\sqrt{(1+2n)(b_l^2\omega - a_l b_l\kappa - a_l)}} \tag{205}$$

which immediately prompts the constraint

$$\eta_l(1-b_l\kappa)(b_l^2\omega - a_l b_l\kappa - a_l) > 0. \tag{206}$$

Thus, we secure dark and singular soliton solutions given by

$$q^{(l)}(x,t) = \left\{-\frac{(1+2n)\xi_l}{4(1+n)\eta_l}\left[1 \pm \tanh\left\{\frac{n\xi_l}{2(1+n)}\sqrt{\frac{(1+2n)(1-b_l\kappa)}{\eta_l(b_l^2\omega - a_l b_l\kappa - a_l)}}(x - vt)\right\}\right]\right\}^{\frac{1}{2n}} \times e^{i(-\kappa x + \omega t + \theta)} \tag{207}$$

and

$$q^{(l)}(x,t) = \left\{-\frac{(1+2n)\xi_l}{4(1+n)\eta_l}\left[1 \pm \coth\left\{\frac{n\xi_l}{2(1+n)}\sqrt{\frac{(1+2n)(1-b_l\kappa)}{\eta_l(b_l^2\omega - a_l b_l\kappa - a_l)}}(x - vt)\right\}\right]\right\}^{\frac{1}{2n}} \times e^{i(-\kappa x + \omega t + \theta)}, \tag{208}$$

respectively, where parameter B is given by (205), while the speed of the soliton is (141) or (201) and the wave number of the soliton is dictated by (204). These solitons will exist provided the constraint condition given by (206) holds.

5. Multiple-Core Couplers (Coupling with All Neighbors)

The governing equation for multiple-core couplers, for coupling with all neighbors is[38]

$$iq_t^{(l)} + a_l q_{xx}^{(l)} + b_l q_{xt}^{(l)} + c_l F(|q^{(l)}|^2) q^{(l)} = \sum_{m=1}^{N} \lambda_{lm} q^{(m)}, \tag{209}$$

where $1 \leq l \leq N$ and λ_{lm} represents the coupling coefficient with all neighbors. The solution hypothesis is taken to be the same as given by (140). Substituting this hypothesis into (209) and again splitting into real and imaginary parts, one obtains the same speed of solitons, as in (141), that is valid for all types of solitons in all nonlinear media considered in this paper. The real part equation now is

$$a_l \frac{\partial^2 P_l}{\partial x^2} + b_l \frac{\partial^2 P_l}{\partial x \partial t} + P_l(b_l \omega \kappa - \omega - a_l \kappa^2) + c_l F(P_l^2) P_l - \sum_{m=1}^{N} \lambda_{lm} P_m = 0. \tag{210}$$

Under the traveling wave transformation

$$P_l(x,t) = U_l(\tau), \quad \tau = B(x - vt) \tag{211}$$

we have

$$(a_l - b_l v) B^2 \frac{d^2 U_l}{d\tau^2} + U_l(b_l \omega \kappa - \omega - a_l \kappa^2) + c_l F(U_l^2) U_l - \sum_{m=1}^{N} \lambda_{lm} U_m = 0. \tag{212}$$

5.1. *Kerr law nonlinearity*

For Kerr law, governing equation is[38]

$$iq_t^{(l)} + a_l q_{xx}^{(l)} + b_l q_{xt}^{(l)} + c_l |q^{(l)}|^2 q^{(l)} = \sum_{m=1}^{N} \lambda_{lm} q^{(m)}. \tag{213}$$

The real part equation (212) therefore reduces to

$$(a_l - b_l v) B^2 \frac{d^2 U_l}{d\tau^2} + U_l(b_l \omega \kappa - \omega - a_l \kappa^2) + c_l U_l^3 - \sum_{m=1}^{N} \lambda_{lm} U_m = 0. \tag{214}$$

We then assume that Eq. (214) has the following formal solution:

$$U_l(\tau) = A_l\left(\frac{G'}{G}\right), \quad A_l \neq 0, \tag{215}$$

where A_l are constants to be determined later and G satisfies Eq. (30). Thus, we obtain

$$(a_l - b_l v)B^2\left\{2A_l\left(\frac{G'}{G}\right)^3 + 3A_l\lambda\left(\frac{G'}{G}\right)^2 + (2A_l\mu + A_l\lambda^2)\left(\frac{G'}{G}\right) + \lambda\mu A_l\right\}$$
$$+ A_l\left(\frac{G'}{G}\right)(b_l\omega\kappa - \omega - a_l\kappa^2) + c_l A_l^3\left(\frac{G'}{G}\right)^3 - \sum_{m=1}^{N}\lambda_{lm}A_m\left(\frac{G'}{G}\right) = 0. \tag{216}$$

Then, equating the coefficient of each power of G'/G to zero, we obtain a system of nonlinear algebraic equations and by solving it, we get

$$v = \frac{2B^2 a_l + c_l A_l^2}{2B^2 b_l}, \tag{217}$$

$$\lambda = 0, \tag{218}$$

$$\omega = \frac{\mu c_l A_l^3 + A_l a_l \kappa^2 + \sum_{m=1}^{N}\lambda_{lm}A_m}{A_l(b_l\kappa - 1)}, \tag{219}$$

where $B,\ \kappa,\ A_l,\ k_l,\ \mu$ are arbitrary constants.

Next, equating the two values of the speed v from the imaginary part equation (141) and real part equation (217) gives the free parameter

$$B = \pm\frac{A_l(b_l\kappa - 1)\sqrt{-A_l c_l}}{\sqrt{\begin{array}{c}2(b_l^2\mu c_l A_l^3 + b_l^2 A_l a_l\kappa^2 + b_l^2\sum_{m=1}^{N}\lambda_{lm}A_m \\ - A_l a_l b_l\kappa(b_l\kappa - 1) - A_l a_l(b_l\kappa - 1))\end{array}}}, \tag{220}$$

which again poses the constraint

$$A_l c_l\left(b_l^2\mu c_l A_l^3 + b_l^2 A_l a_l\kappa^2 + b_l^2\sum_{n=1}^{N}\lambda_{lm}A_m - A_l a_l b_l\kappa(b_l\kappa - 1) - A_l a_l(b_l\kappa - 1)\right) < 0. \tag{221}$$

Substituting the general solutions of the second-order LODE into the formulas (215), we have three types of traveling wave solutions of Eq. (213).

When $\mu < 0$, we obtain the hyperbolic function traveling wave solution

$$q^{(l)}(x,t) = A_l\sqrt{-\mu}\left(\frac{C_1\sinh(\sqrt{-\mu}(x - vt)) + C_2\cosh(\sqrt{-\mu}(x - vt))}{C_1\cosh(\sqrt{-\mu}(x - vt)) + C_2\sinh(\sqrt{-\mu}(x - vt))}\right) \times e^{i(-\kappa x + \omega t + \theta)}, \tag{222}$$

where C_1 and C_2 are arbitrary constants and parameter B is given by (220), while the speed of the soliton is (141) or (217) and finally the wave number of the soliton is dictated by (219). These solitons will exist provided the constraint condition given by (221) holds. On the other hand, assuming $C_1 \neq 0$ and $C_2 = 0$ dark 1-soliton solution is given by

$$q^{(l)}(x,t) = A_l\sqrt{-\mu}\tanh\{\sqrt{-\mu}(x-vt)\}e^{i(-\kappa x+\omega t+\theta)}, \tag{223}$$

and again assuming $C_1 = 0$ and $C_2 \neq 0$, singular 1-soliton solution for Eq. (213) is

$$q^{(l)}(x,t) = A_l\sqrt{-\mu}\coth\{\sqrt{-\mu}(x-vt)\}e^{i(-\kappa x+\omega t+\theta)}, \tag{224}$$

where parameter B is given by (220), while the speed of the soliton is (141) or (217) and finally the wave number of the soliton is dictated by (219). These solitons will exist as long as the constraint condition given by (221) holds.

If $\mu > 0$, solutions in terms of trigonometric functions are

$$q^{(l)}(x,t) = A_l\sqrt{\mu}\left(\frac{-C_1\sin(\sqrt{\mu}(x-vt)) + C_2\cos(\sqrt{\mu}(x-vt))}{C_1\cos(\sqrt{\mu}(x-vt)) + C_2\sin(\sqrt{\mu}(x-vt))}\right) \times e^{i(-\kappa x+\omega t+\theta)}, \tag{225}$$

where C_1 and C_2 are arbitrary constants and parameter B is given by (220), while the speed of the soliton is (141) or (217) and finally the wave number of the soliton is dictated by (219). These solitons will exist provided the constraint condition given by (221) stays valid.

Also, with the assumption $C_1 \neq 0$ and $C_2 = 0$,

$$q^{(l)}(x,t) = -A_l\sqrt{\mu}\tan\{\sqrt{\mu}(x-vt)\}e^{i(-\kappa x+\omega t+\theta)}, \tag{226}$$

and when $C_1 = 0,\ C_2 \neq 0$ Eq. (213) will be

$$q^{(l)}(x,t) = A_l\sqrt{\mu}\cot\{\sqrt{\mu}(x-vt)\}e^{i(-\kappa x+\omega t+\theta)}, \tag{227}$$

which represents a pair of singular periodic solutions. The parameter B is given by (220), while the speed of the soliton is (141) or (217) and finally the wave number of the soliton is dictated by (219). These solitons will exist provided the constraint condition given by (221) is carried out.

When $\mu = 0$, the plane wave solution is

$$q^{(l)}(x,t) = \frac{A_l C_2}{C_1 + C_2 B(x-vt)}e^{i(-\kappa x+\omega t+\theta)}, \tag{228}$$

with parameter B given by (220), while the speed of the soliton by (141) or (217) and finally the wave number of the soliton is dictated by (219). These solitons will exist provided the constraint condition given by (221) holds.

5.2. *Power law nonlinearity*

For power law, the coupled NLSE modifies to[38]

$$iq_t^{(l)} + a_l q_{xx}^{(l)} + b_l q_{xt}^{(l)} + c_l|q^{(l)}|^{2n}q^{(l)} = \sum_{m=1}^{N}\lambda_{lm}q^{(m)}. \tag{229}$$

Therefore, the real part equation reduces to

$$(a_l - b_l v)B^2 \frac{d^2 U_l}{d\tau^2} + U_l(b_l \omega\kappa - \omega - a_l\kappa^2) + c_l U_l^{2n+1} - \sum_{m=1}^{N} \lambda_{lm} U_m = 0. \quad (230)$$

We then assume that Eq. (230) has the following formal solution:

$$U_l(\tau) = A_l \left(\frac{G'}{G}\right)^{\frac{1}{n}}, \quad A_l \neq 0, \quad (231)$$

where A_l are constants to be determined later and G satisfies Eq. (30). Thus, we obtain

$$\begin{aligned}
&(a_l - b_l v)B^2 \left\{ \left(\frac{1}{n^2} + \frac{1}{n}\right) A_l \left(\frac{G'}{G}\right)^{\frac{1}{n}+2} + \left(\frac{2}{n^2} + \frac{1}{n}\right) A_l \lambda \left(\frac{G'}{G}\right)^{\frac{1}{n}+1} \right. \\
&\quad + \left(\frac{2}{n^2} A_l \mu + \frac{1}{n^2} A_l \lambda^2\right) \left(\frac{G'}{G}\right)^{\frac{1}{n}} + \left(\frac{2}{n^2} - \frac{1}{n}\right) A_l \mu\lambda \left(\frac{G'}{G}\right)^{\frac{1}{n}-1} \\
&\quad \left. + \left(\frac{1}{n^2} - \frac{1}{n}\right) A_l \mu^2 \left(\frac{G'}{G}\right)^{\frac{1}{n}-2} \right\} \\
&\quad + A_l \left(\frac{G'}{G}\right)^{\frac{1}{n}} (b_l \omega\kappa - \omega - a_l \kappa^2) + c_l A_l^{2n+1} \left(\frac{G'}{G}\right)^{\frac{1}{n}+2} \\
&\quad - \sum_{m=1}^{N} \lambda_{lm} A_m \left(\frac{G'}{G}\right)^{\frac{1}{n}} = 0. \quad (232)
\end{aligned}$$

Then, equating the coefficient of each power of G'/G to zero, we obtain a system of nonlinear algebraic equations and by solving it, we get

$$v = \frac{(1+n)B^2 a_l + c_l n^2 A_l^{2n}}{(1+n)B^2 b_l}, \quad (233)$$

$$\lambda = \mu = 0, \quad (234)$$

$$\omega = \frac{A_l a_l \kappa^2 + \sum_{m=1}^{N} \lambda_{lm} A_m}{A_l(b_l\kappa - 1)}, \quad (235)$$

where $B,\ \kappa,\ A_l,\ k_l$ are arbitrary constants. Next, equating the two values of the speed v from the imaginary part equation (141) and real part equation (233) gives the free parameter

$$B = \pm \frac{n A_l^n (b_l\kappa - 1)\sqrt{-A_l c_l}}{\sqrt{(1+n)(b_l^2 A_l a_l \kappa^2 + b_l^2 \sum_{m=1}^{N} \lambda_{lm} A_m - A_l a_l b_l \kappa(b_l\kappa - 1) - A_l a_l (b_l\kappa - 1))}}, \quad (236)$$

which again poses the constraint

$$A_l c_l \left(b_l^2 A_l a_l \kappa^2 + b_l^2 \sum_{m=1}^{N} \lambda_{lm} A_m - A_l a_l b_l \kappa(b_l\kappa - 1) - A_l a_l (b_l\kappa - 1) \right) < 0. \quad (237)$$

Since $\lambda = \mu = 0$, we obtain plane wave solution

$$q^{(l)}(x,t) = A_l\left\{\frac{C_2}{C_1 + C_2 B(x - vt)}\right\}^{\frac{1}{n}} e^{i(-\kappa x + \omega t + \theta)}, \tag{238}$$

where C_1 and C_2 are arbitrary constants and parameter B is given by (236), while the speed of the soliton is (141) or (233) and finally the wave number of the soliton is dictated by (235). These solitons will exist with the constraint condition given by (237).

On the other hand, to obtain an analytic solution, we use the transformations $U_l = V_l^{\frac{1}{2n}}$, $U_1 = U_2 = \cdots = U_N = U_l$ in Eq. (230) to find

$$(a_l - b_l v)B^2\{(1-2n)(V_l')^2 + 2nV_lV_l''\} + 4(b_l\omega\kappa - \omega - a_l\kappa^2)n^2V_l^2$$
$$+ 4c_ln^2V_l^3 - 4n^2\sum_{m=1}^{N}\lambda_{lm}V_l^2 = 0. \tag{239}$$

Balancing the order of V_lV_l'' and V_l^3 in Eq. (239), we have $N = 2$. Therefore, Eq. (239) has the solutions in the form

$$V_l(\tau) = A_{l0} + A_{l1}\left(\frac{G'}{G}\right) + A_{l2}\left(\frac{G'}{G}\right)^2. \tag{240}$$

Substituting Eq. (240) along with Eq. (30) in Eq. (239) and setting all the coefficients of powers G'/G to be zero, then we obtain a system of nonlinear algebraic equations and by solving it, we obtain

$$\omega = \frac{B^2(b_lv - a_l)(\lambda^2 - 4\mu) + 4n^2(a_l\kappa^2 + \sum_{m=1}^{N}\lambda_{lm})}{4n^2(b_l\kappa - 1)}, \tag{241}$$

$$A_{l0} = -\frac{\mu B^2(a_l - b_lv)(1+n)}{n^2c_l}, \tag{242}$$

$$A_{l1} = -\frac{\lambda B^2(a_l - b_lv)(1+n)}{n^2c_l}, \tag{243}$$

$$A_{l2} = -\frac{B^2(a_l - b_lv)(1+n)}{n^2c_l}, \tag{244}$$

where $B,\ \kappa,\ \lambda,\ \mu$ are arbitrary constants.

Now, from Eqs. (141) and (241), we have

$$\omega = \frac{B^2a_l(b_l\kappa + 1)(\lambda^2 - 4\mu) + (b_l\kappa - 1)(4n^2a_l\kappa^2 + 4n^2\sum_{m=1}^{N}\lambda_{lm})}{4n^2(b_l\kappa - 1)^2 + (\lambda^2 - 4\mu)B^2b_l^2}. \tag{245}$$

Substituting the general solutions of second-order LODE into (240) leads to three types of traveling wave solutions.

When $\Delta = \lambda^2 - 4\mu > 0$, we obtain the hyperbolic function traveling wave solution

$$q^{(l)}(x,t) = \left[\frac{B^2(a_l - b_l v)(\lambda^2 - 4\mu)(1+n)}{4n^2 c_l} \times \left(1 - \left(\frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}B(x-vt)\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}B(x-vt)\right)}{C_1 \cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}B(x-vt)\right) + C_2 \sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}B(x-vt)\right)}\right)^2\right)\right]^{\frac{1}{2n}} \times e^{i(-\kappa x + \omega t + \theta)}, \tag{246}$$

where C_1 and C_2 are arbitrary constants and ω is given by Eq. (245), while v is given by Eq. (141).

As a special case, assuming $C_1 \neq 0$ and $C_2 = 0$ one secures dark 1-soliton solution

$$q^{(l)}(x,t) = \left[\frac{B^2(a_l - b_l v)(\lambda^2 - 4\mu)(1+n)}{4n^2 c_l} \tanh^2\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}B(x-vt)\right\}\right]^{\frac{1}{2n}} \times e^{i(-\kappa x + \omega t + \theta)}, \tag{247}$$

while if $C_1 = 0$ and $C_2 \neq 0$, the singular 1-soliton solution is

$$q^{(l)}(x,t) = \left[-\frac{B^2(a_l - b_l v)(\lambda^2 - 4\mu)(1+n)}{4n^2 c_l} \coth^2\left\{\frac{\sqrt{\lambda^2-4\mu}}{2}B(x-vt)\right\}\right]^{\frac{1}{2n}} \times e^{i(-\kappa x + \omega t + \theta)}, \tag{248}$$

with ω given by Eq. (245), while v given by Eq. (141).

With $\Delta = \lambda^2 - 4\mu < 0$, we obtain traveling wave solution with trigonometric functions

$$q^{(l)}(x,t) = \left[\frac{B^2(a_l - b_l v)(\lambda^2 - 4\mu)(1+n)}{4n^2 c_l} \times \left(1 + \left(\frac{-C_1 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}B(x-vt)\right) + C_2 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}B(x-vt)\right)}{C_1 \cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}B(x-vt)\right) + C_2 \sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}B(x-vt)\right)}\right)^2\right)\right]^{\frac{1}{2n}} \times e^{i(-\kappa x + \omega t + \theta)}, \tag{249}$$

where C_1 and C_2 are arbitrary constants and ω is given by Eq. (245), while v is given by Eq. (141).

For the special case, $C_1 \neq 0$ and $C_2 = 0$,

$$q^{(l)}(x,t) = \left[\frac{B^2(a_l - b_l v)(\lambda^2 - 4\mu)(1+n)}{4n^2 c_l}\tan^2\left\{\frac{\sqrt{4\mu - \lambda^2}}{2}B(x - vt)\right\}\right]^{\frac{1}{2n}} \times e^{i(-\kappa x + \omega t + \theta)}, \quad (250)$$

and $C_1 = 0,\ C_2 \neq 0$

$$q^{(l)}(x,t) = \left[\frac{B^2(a_l - b_l v)(\lambda^2 - 4\mu)(1+n)}{4n^2 c_l}\cot^2\left\{\frac{\sqrt{4\mu - \lambda^2}}{2}B(x - vt)\right\}\right]^{\frac{1}{2n}} \times e^{i(-\kappa x + \omega t + \theta)}, \quad (251)$$

which represents singular periodic solutions, where ω is given by Eq. (245), while v is given by Eq. (141).

Finally, for $\Delta = \lambda^2 - 4\mu = 0$, the plane wave solution is

$$q^{(l)}(x,t) = \left[-\frac{B^2(a_l - b_l v)(1+n)}{n^2 c_l}\left\{\frac{C_2}{C_1 + C_2 B(x - vt)}\right\}^2\right]^{\frac{1}{2n}} \times e^{i(-\kappa x + \omega t + \theta)}, \quad (252)$$

where ω is given by Eq. (245), while v is given by Eq. (141).

5.3. *Parabolic law nonlinearity*

In this case, the governing equation reduces to[38]

$$iq_t^{(l)} + a_l q_{xx}^{(l)} + b_l q_{xt}^{(l)} + (\xi_l |q^{(l)}|^2 + \eta_l |q^{(l)}|^4) q^{(l)} = \sum_{m=1}^{N} \lambda_{lm} q^{(m)}, \quad (253)$$

where $1 \leq l \leq N$. The real part equation therefore is

$$(a_l - b_l v)B^2 \frac{d^2 U_l}{d\tau^2} + U_l(b_l \omega\kappa - \omega - a_l \kappa^2) + \xi_l U_l^3 + \eta_l U_l^5 - \sum_{m=1}^{N} \lambda_{lm} U_m = 0. \quad (254)$$

We then assume that Eq. (254) has the following formal solutions:

$$U_l(\tau) = A_l \left(\frac{G'}{G}\right)^{\frac{1}{2}}, \quad A_l \neq 0, \quad (255)$$

where A_l are constants to be determined later and G satisfies Eq. (30). Then,

$$(a_l - b_l v)B^2 \left\{\frac{3}{4}A_l \left(\frac{G'}{G}\right)^{\frac{5}{2}} + A_l \lambda \left(\frac{G'}{G}\right)^{\frac{3}{2}} + \left(\frac{1}{2}A_l \mu + \frac{1}{4}A_l \lambda^2\right)\left(\frac{G'}{G}\right)^{\frac{1}{2}} - \frac{1}{4}A_l \mu^2 \left(\frac{G'}{G}\right)^{-\frac{3}{2}}\right\}$$

$$+A_l\left(\frac{G'}{G}\right)^{\frac{1}{2}}(b_l\omega\kappa-\omega-a_l\kappa^2)+\xi_l\left\{A_l\left(\frac{G'}{G}\right)^{\frac{1}{2}}\right\}^3+\eta_l\left\{A_l\left(\frac{G'}{G}\right)^{\frac{1}{2}}\right\}^5$$

$$-\sum_{m=1}^{N}\lambda_{lm}A_m\left(\frac{G'}{G}\right)^{\frac{1}{2}}=0. \tag{256}$$

Then, equating the coefficient of each power of G'/G to zero, we obtain a system of nonlinear algebraic equations and by solving it, we get

$$v=\frac{4\eta_l A_l^4+3B^2a_l}{3B^2b_l}, \tag{257}$$

$$\lambda=\frac{3\xi_l}{4A_l^2\eta_l}, \tag{258}$$

$$\mu=0, \tag{259}$$

$$\omega=\frac{3\xi_l^2A_l+16A_la_l\kappa^2\eta_l+16\eta_l\sum_{m=1}^{N}\lambda_{lm}A_m}{16A_l\eta_l(b_l\kappa-1)}, \tag{260}$$

where $B,\ \kappa,\ A_l,\ k_l$ are arbitrary constants.

Next, equating the two values of the speed v from the imaginary part equation (141) and real part equation (257) gives the free parameter

$$B=\pm\frac{2A_l^2\sqrt{\eta_l(1-b_l\kappa)}}{\sqrt{3(b_l^2\omega-a_lb_l\kappa-a_l)}} \tag{261}$$

which immediately kicks in the constraint

$$\eta_l(1-b_l\kappa)(b_l^2\omega-a_lb_l\kappa-a_l)>0. \tag{262}$$

Thus, we obtain the exact traveling wave solution of Eq. (253) as

$$q^{(l)}(x,t)=\left\{-\frac{3\xi_l}{8\eta_l}\left[1\pm\tanh\left\{\frac{\xi_l}{4}\sqrt{\frac{3(1-b_l\kappa)}{\eta_l(b_l^2\omega-a_lb_l\kappa-a_l)}}(x-vt)\right\}\right]\right\}^{\frac{1}{2}}\times e^{i(-\kappa x+\omega t+\theta)} \tag{263}$$

and

$$q^{(l)}(x,t)=\left\{-\frac{3\xi_l}{8\eta_l}\left[1\pm\coth\left\{\frac{\xi_l}{4}\sqrt{\frac{3(1-b_l\kappa)}{\eta_l(b_l^2\omega-a_lb_l\kappa-a_l)}}(x-vt)\right\}\right]\right\}^{\frac{1}{2}}\times e^{i(-\kappa x+\omega t+\theta)}, \tag{264}$$

which are dark and singular soliton solutions, respectively. The parameter B is given by (261), while the speed of the soliton is (141) or (257) and the wave number of the soliton is dictated by (260). These solitons will exist provided the constraint condition given by (262) holds.

5.4. *Dual-power law nonlinearity*

For dual-power law nonlinearity, the governing equation is[38]

$$iq_t^{(l)} + a_l q_{xx}^{(l)} + b_l q_{xt}^{(l)} + (\xi_l |q^{(l)}|^{2n} + \eta_l |q^{(l)}|^{4n}) q^{(l)} = \sum_{m=1}^{N} \lambda_{lm} q^{(m)}, \tag{265}$$

where $1 \le l \le N$. The real part equation therefore is

$$(a_l - b_l v)B^2 \frac{d^2U_l}{d\tau^2} + U_l(b_l\omega\kappa - \omega - a_l\kappa^2) + \xi_l U_l^{2n+1} + \eta_l U_l^{4n+1} - \sum_{m=1}^{N} \lambda_{lm} U_m = 0, \tag{266}$$

We then assume that Eq. (265) has the following formal solutions:

$$U_l(\tau) = A_l \left(\frac{G'}{G}\right)^{\frac{1}{2n}}, \quad A_l \neq 0, \tag{267}$$

where A_l are constants to be determined later and G satisfies Eq. (30). Thus,

$$\begin{aligned}
&(a_l - b_l v)B^2 \Bigg\{ \left(\frac{1}{4n^2} + \frac{1}{2n}\right) A_l \left(\frac{G'}{G}\right)^{\frac{1}{2n}+2} + \left(\frac{1}{2n^2} + \frac{1}{2n}\right) A_1 \lambda \left(\frac{G'}{G}\right)^{\frac{1}{2n}+1} \\
&\quad + \left(\frac{1}{2n^2} A_l \mu + \frac{1}{4n^2} A_l \lambda^2\right) \left(\frac{G'}{G}\right)^{\frac{1}{2n}} + \left(\frac{1}{2n^2} - \frac{1}{2n}\right) A_l \mu\lambda \left(\frac{G'}{G}\right)^{\frac{1}{2n}-1} \\
&\quad + \left(\frac{1}{4n^2} - \frac{1}{2n}\right) A_l \mu^2 \left(\frac{G'}{G}\right)^{\frac{1}{2n}-2} \Bigg\} \\
&\quad + A_l \left(\frac{G'}{G}\right)^{\frac{1}{2n}} (b_l\omega\kappa - \omega - a_l\kappa^2) + \xi_l \left\{ A_l \left(\frac{G'}{G}\right)^{\frac{1}{2n}} \right\}^{2n+1} \\
&\quad + \eta_l \left\{ A_l \left(\frac{G'}{G}\right)^{\frac{1}{2n}} \right\}^{4n+1} - \sum_{m=1}^{N} \lambda_{lm} A_m \left(\frac{G'}{G}\right)^{\frac{1}{2n}} = 0.
\end{aligned} \tag{268}$$

Then, equating the coefficient of each power of G'/G to zero, we obtain a system of nonlinear algebraic equations and by solving it, we get

$$v = \frac{4n^2\eta_l A_l^{4n} + (1+2n)B^2 a_l}{(1+2n)B^2 b_l}, \tag{269}$$

$$\lambda = \frac{(1+2n)\xi_l}{2(1+n)A_l^{2n}\eta_l}, \tag{270}$$

$$\mu = 0, \tag{271}$$

$$\omega = \frac{(1+2n)\xi_l^2 A_l + 4(1+n)^2 A_l a_l \kappa^2 \eta_l + 4(1+n)^2 \eta_l \sum_{m=1}^{N} \lambda_{lm} A_m}{4(1+n)^2 A_l \eta_l (b_l\kappa - 1)}, \tag{272}$$

where $B,\ \kappa,\ A_l,\ k_l$ are arbitrary constants.

Next, equating the two values of the speed v from the imaginary part equation (141) and real part equation (269) gives the free parameter

$$B = \pm \frac{2nA_l^{2n}\sqrt{\eta_l(1-b_l\kappa)}}{\sqrt{(1+2n)(b_l^2\omega - a_lb_l\kappa - a_l)}} \tag{273}$$

which immediately introduces the constraint

$$\eta_l(1-b_l\kappa)(b_l^2\omega - a_lb_l\kappa - a_l) > 0. \tag{274}$$

Thus, we obtain the exact traveling wave solution of Eq. (265) as

$$q^{(l)}(x,t) = \left\{-\frac{(1+2n)\xi_l}{4(1+n)\eta_l}\left[1 \pm \tanh\left\{\frac{n\xi_l}{2(1+n)}\sqrt{\frac{(1+2n)(1-b_l\kappa)}{\eta_l(b_l^2\omega - a_lb_l\kappa - a_l)}}(x-vt)\right\}\right]\right\}^{\frac{1}{2n}} \times e^{i(-\kappa x+\omega t+\theta)} \tag{275}$$

and

$$q^{(l)}(x,t) = \left\{-\frac{(1+2n)\xi_l}{4(1+n)\eta_l}\left[1 \pm \coth\left\{\frac{n\xi_l}{2(1+n)}\sqrt{\frac{(1+2n)(1-b_l\kappa)}{\eta_l(b_l^2\omega - a_lb_l\kappa - a_l)}}(x-vt)\right\}\right]\right\}^{\frac{1}{2n}} \times e^{i(-\kappa x+\omega t+\theta)}, \tag{276}$$

which represents dark and singular 1-soliton solutions, respectively. The parameter B is given by (273), while the speed of the soliton is (141) or (269) and the wave number of the soliton is dictated by (272). These solitons will exist provided the constraint condition given by (274) are in place.

6. Conclusions

This paper obtained dark and singular soliton solutions to optical couplers by using G'/G-expansion method. There are four nonlinear media studied in this paper. They are Kerr law, power law, parabolic law, and dual-power law. Also, two types of couplers are considered, which are twin-core couplers and multiple-core couplers. This integration scheme, applied to all of these four laws of nonlinearity, only allows to retrieve dark and singular solitons for all laws of nonlinearity and both forms of couplers. This therefore presents a limitation to this scheme being unable to obtain bright soliton solutions. However, bright soliton solutions, for optical couplers are already reported during 2014 with ansatz approach.[38]

There is another drawback with this integration scheme, this tool cannot be applied to couplers with log-law nonlinearity. Therefore, the discussion on couplers in log-law nonlinear medium is skipped in this paper. However, exact 1-soliton solutions for log-law medium, known as *Gaussons* were reported in the past where ansatz approach was implemented to extract these Gaussons.[8,38]

The results of this paper is a pillar of strength for further studies in the area of optical couplers. It is therefore necessary to address the governing NLSE for optical couplers with several perturbation terms. This will lead to an extended version of NLSE which will later be integrated and the results will be reported soon. Moreover, additional integration techniques such as Lie symmetry will be implemented to integrate the model equation. Additionally, the governing equation will be considered with time-dependent coefficients. All of these exciting projects will be handled shortly.

Acknowledgments

This research is funded by Qatar National Research Fund (QNRF) under the grant number NPRP 6-021-1-005. The sixth and seventh authors (AB & MB) thankfully acknowledge this support from QNRF. The authors declare that there is no conflict of interest.

References

1. A. A. Alshaery, E. M. Hilal, M. A. Banaja, S. A. Alkhateeb, L. Moraru and A. Biswas, Optical solitons in multiple-core couplers, *J. Optoelectron. Adv. Mater.* **16**(5–6) (2014) 750–758.
2. A. Biswas. Solitons in nonlinear fiber arrays, *J. Electromagn. Waves Appl.* **15**(9) (2001) 1189–1196.
3. A. Biswas, Solitons in multiple-core couplers, *J. Nonlinear Opt. Phys. Mater.* **10**(3) (2001) 329–336.
4. A. Biswas, Dispersion-managed solitons in nonlinear fiber arrays, *Fiber Integr. Opt.* **20**(6) (2001) 521–579.
5. A. Biswas, Theory of optical couplers, *Opt. Quant. Electron.* **35**(3) (2003) 221–235.
6. A. Biswas, Dispersion-managed solitons in optical couplers, *J. Nonlinear Opt. Phys Mater.* **12**(1) (2003) 45–74.
7. A. Biswas, D. Milovic, M. Savescu, M. F. Mahmood, K. R. Khan and R. Kohl, Optical soliton perturbation in nanofibers with improved nonlinear Schrödinger's equation by semi-inverse variational principle, *J. Nonlinear Opt. Phys. Mater.* **12**(4) (2012) 1250054.
8. A. Biswas, D. A. Lott, B. Sutton, K. R. Khan and M. F. Mahmood, Optical Gaussons in nonlinear directional couplers, *J. Electromagn. Waves Appl.* **27**(15) (2013) 1976–1985.
9. M. Chen, Z. Liu, R. Lu, Y. Yang, M. Liang and T.-T. Tang, An ultracompact optical directional coupler based on lithium niobate photonic wires, *Optik* **124**(14) (2013) 1974–1976.
10. M. Eslami, M. Mirzazadeh and A. Biswas, Soliton solutions of the resonant nonlinear Schrödinger's equation in optical fibers with time-dependent coefficients by simplest equation approach, *J. Mod. Opt.* **60**(19) (2013) 1627–1636.
11. M. Eslami, M. Mirzazadeh, B. F. Vajargah and A. Biswas, Optical solitons for the resonant nonlinear Schrödinger's equation with time-dependent coefficients by the first integral method, *Optik.* **125**(9) (2014) 3107–3116.

12. Y. Fang and J. Zhou, Effects of third-order dispersion on soliton switching in fiber nonlinear directional couplers, *Optik.* **119**(2) (2008) 86–89.
13. X. Geng and Y. Lv, Darboux transformation for an integrable generalization of the nonlinear Schrödinger equation, *Nonlinear Dyn.* **69**(4) (2012) 1621–1630.
14. A. Govindaraji, A. Mahalingam and Uthayakumar, Dark soliton switching in nonlinear fiber couplers with gain, *Opt. Laser Technol.* **60** (2014) 18–21.
15. P. Green, D. Milovic, A. K. Sarma, D. Lott and A. Biswas, Dynamics of super-sech solitons in optical fibers, *J. Nonlinear Opt. Phys. Mater.* **19**(2) (2010) 339–370.
16. H. He and L. Wang, Numerical analysis of birefringence and coupling length on dual-core photonics crystal fiber with complex air holes, *Optik.* **124**(23) (2013) 5941–5944.
17. X. He, K. Xie and H. Yang, Gain-induced soliton switching in fiber nonlinear directional coupler, *Optik.* **123**(24) (2012) 2247–2249.
18. D. Irawan, Saktioto, J. Ali and M. Fadhali, Birefringence analysis of directional fiber coupler induced by fusion and coupling parameters, *Opt.* **124**(17) (2013) 3063–3066.
19. K. R. Khan and T. Wu, Short pulse propagation in wavelength selective index guided photonic crystal fiber coupler, *IEEE J. Sel. Topics Quant. Electron.* **14** (2008) 752–757.
20. K. R. Khan, T. Wu, D. N. Christodoulides and G. I. Stegeman, Soliton switching and multi-frequency generation in nonlinear photonic crystal fiber, *Opt. Express* **16**(13) (2008) 9417–9428.
21. S. Kumar, K. Singh and R. K. Gupta, Coupled Higgs field equation and Hamiltonian amplitude equation: Lie classical approach and G'/G- expansion method, *Pramana* **79**(1) (2012) 41–60.
22. L. Lenells and A. S. Fokas, Dressing for a novel integrable generalization of the nonlinear Schrödinger's equation, *J. Nonlinear Sci.* **20** (2010) 709–722.
23. J. Li, J. Cao and X. Xu, Effects of phase errors on phase locking of all-fiber laser arrays, *Opt. Laser Technol.* **47** (2013) 372–378.
24. H. Li, X. Dong, E. Li, Z. Liu and Y. Bai, Highly compact 2×2 multiport interference coupler in silicon photonic nanowires for array waveguide grating demodulation integration microsystem, *Opt. Laser Technol.* **47** (2013) 366–371.
25. L. Li and G. Q. Liu, Photonic crystal ring resonator channel drop filter, *Optik.* **124**(17) (2013) 2966–2968.
26. Y. Li, M. Jiang, C. Zhang and Y. Sun, Transmitting spectra-temperature characteristic of single mode fiber coupler for fiber gyro application, *Optik.* **124**(23) (2013) 6326–6329.
27. P. Mandal and S. Midda, All optical method of developing OR and NAND logic system based on nonlinear fiber couplers, *Optik.* **122**(20) (2011) 1795–1798.
28. P. Mandal, Method of developing all optical half-adder based on nonlinear directional coupler, *Opt. Photonics Lett.* **6**(1) (2013) 1350001.
29. M. Mirzazadeh, M. Eslami, D. Milovic and A. Biswas, Topological solitons of resonant nonliner Schrödinger's equation with dual-power law nonlinerity using G'/G-expansion technique, *Optik.* **125**(19) (2014) 5480–5489.
30. M. Mirzazadeh, M. Eslami, B. F. Vajargah and A. Biswas, Optical solitons and optical rogons of generalized resonant dispersive nonlinear Schrödinger's equation with power law nonlinearity, *Optik.* **125**(9) (2014) 4246–4256.
31. M. Mirzazadeh, M. Eslami, E. Zerrad, M. F. Mahmood, A. Biswas and M. Belic, Optical solitons in nonlinear directional couplers by sine-cosine function method and Bernoulli's equation approach, To appear in *Nonlinear Dynamics* DOI: 10.1007/s11071-0815-2117-y.

32. M. J. Potasek and Y. Yang, Multiterabit-per-second all-optical switching in a nonlinear directional coupler, *IEEE J. Sel. Topics Quantum Electron.* **8** (2002) 714–721.
33. S. Pu, C. Hou and C. Yuan. Soliton switching in inhomogeneous nonlocal media, *Optik.* **125**(3) (2014) 1075–1078.
34. T. Pustelny and P. Struk, Numerical analyses of optical couplers for planar waveguides, *Optoelectron. Lett.* **20**(3) (2012) 201–206.
35. A. K. Sarma, A comparitive study of soliton switching in two- and three-core coupler with TOD and IMD, *Optik.* **120**(8) (2009) 390–394.
36. A. K. Sarma, Dark soliton switching in an NLDC in presence of higher order perturbative effects, *Opt. Laser Technol.* **41**(3) (2009) 247–250.
37. A. K. Sarma, Vector soliton switching in a fiber nonlinear directional coupler, *Opt. Commun.* **284**(1) (2011) 186–190.
38. M. Savescu, A. H. Bhrawy, A. A. Alshaery, E. M. Hilal, K. R. Khan, M. F. Mahmood and A. Biswas, Optical soliton in nonlinear directional couplers with spatio-temporal dispersion, *J. Modern Opt.* **61**(5) (2014) 441–458.
39. Q. Xu, K. Xie, J. Tang and J. Ping, Directional coupler design based on coupled cavity waveguides in photonic crystals, *Optik.* **122**(13) (2011) 1132–1135.
40. Q. Zhou, Q. Zhou, Y. Liu, H. Yu, C. Wei, P. Yao, A. H. Bhrawy and A. Biswas, Bright, dark and singular optical solitons in cascaded system, *Laser Phys.* **25**(2) (2015) 025402.