Journal of Nonlinear Optical Physics & Materials
Vol. 26, No. 1 (2017) 1750005 (30 pages)

DOI: 10.1142/S0218863517500059

Nematicons in liquid crystals by extended trial equation method

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Received 10 December 2016

This paper employs extended trial equation method to retrieve nematicons in liquid crystals from its governing equation. In addition, several other forms of solution naturally emerged from the integration algorithm. These are shock waves, singular solitons, snoidal waves, periodic singular waves, plane waves and others. These variety of solutions are being reported for the first time in the context of liquid crystals.

Keywords: Nematicons; liquid crystals; extended trail equation method.

1. Introduction

While solitons in nonlinear optical fibers, crystals, metamaterials and metasurfaces are well studied and have received a lot of attention, it is equally important to study these soliton molecules in liquid crystals.[1–21] Here, these solitons have a special name which is *nematicons* that was first coined by Assanto.[1–9] This paper obtains exact solutions to the governing model to retrieve nematicons. The methodology, adopted in this paper, is the extended trial equation.[22–26] There are a variety of integration schemes that are applied to solve such nonlinear evolution equations.[27–32] It must be noted that exact expressions for nematicons were not available till 2015.[21] The model was therefore studied, till then, numerically as well as analytically using the variational principle, where a hypothesis for these nematicons was picked.[1–14] The exact expression for nematicons along with its conserved quantities was first obtained by the method of undetermined coefficients during 2015.[21] There are four laws of nonlinearity considered in this paper. They are Kerr law, power law, parabolic and dual-power law. This paper details, in the next four sections, extraction of solitons and other solutions in liquid crystals. These additional forms of waves, obtained in this paper, are plane waves, shock waves, singular solitons, singular periodic waves and others.

1.1. *Governing equation*

The dynamics of solitons in liquid crystals, in dimensionless form, is governed by the following coupled system of equations[1–21]:

$$iq_t + aq_{xx} + brq = 0, \tag{1}$$

$$cr_{xx} + \lambda r + \alpha F(|q|^2) = 0. \tag{2}$$

In (1) and (2), the variables $q(x,t)$ represent the wave profile, while the second dependent variable $r(x,t)$ is the angle of tilt of the liquid crystal molecule. The first term in (1) represents temporal evolution of nematicons, while the second term is the group velocity dispersion (GVD). Also, α is the coefficient of nonlinear term. The functional F is the type of nonlinearity that will be studied. Moreover, a, b, c, λ and α are all constants.

In order to analyze and solve equation pair (1) and (2), the solution hypothesis chosen in phase-amplitude format is

$$q(x,t) = P(\xi)\exp(i\phi), \tag{3}$$

and

$$r(x,t) = Q(\xi), \tag{4}$$

where $\xi = B(x - vt)$ and the phase $\phi(x,t)$ is given by

$$\phi = -\kappa x + \omega t + \theta. \tag{5}$$

Here, κ is the soliton frequency, while ω is the wave number of the soliton and θ is a phase constant. Substituting (3) and (4) into (1) and (2) and then decomposing into real and imaginary parts gives

$$-(v + 2a\kappa)B\frac{dP}{d\xi} = 0 \tag{6}$$

from the imaginary part that leads to the speed of the soliton being

$$v = -2a\kappa. \tag{7}$$

Next, the real part equation simplifies to

$$-(\omega + a\kappa^2)P + aB^2\frac{d^2P}{d\xi^2} + bPQ = 0, \tag{8}$$

while (2) leads to

$$cB^2\frac{d^2Q}{d\xi^2} + \lambda Q + \alpha F(P^2) = 0. \tag{9}$$

Soliton solutions will now be derived for four forms of nonlinearity for the functions F. These are sequentially discussed in the following sections.

2. Kerr Law

This is the simplest form of nonlinearity that is studied in the context of nonlinear optics. In this case, the refractive index of light is intensity-dependent. This formulates the so-called Kerr law. For such law,

$$F(s) = s. \tag{10}$$

Equation (2) for Kerr law nonlinearity reduces to

$$cr_{xx} + \lambda r + \alpha |q|^2 = 0. \tag{11}$$

In this case, Eq. (9) simplifies to

$$cB^2 Q'' + \lambda Q + \alpha P^2 = 0. \tag{12}$$

By extended trial equation method,[22–26] this section will retrieve bright, dark and singular soliton solutions in liquid crystals that are governed by Eqs. (1) and (11).

To start with the extraction of solutions to (8) and (12), the following assumption for the soliton structure is made:

$$P = \sum_{i=0}^{\varsigma} \tau_i \Psi^i, \tag{13}$$

$$Q = \sum_{i=0}^{\tilde{\varsigma}} \tilde{\tau}_i \Psi^i, \tag{14}$$

where

$$(\Psi')^2 = \Lambda(\Psi) = \frac{\Phi(\Psi)}{\Upsilon(\Psi)} = \frac{\mu_\sigma \Psi^\sigma + \cdots + \mu_1 \Psi + \mu_0}{\chi_\rho \Psi^\rho + \cdots + \chi_1 \Psi + \chi_0}. \tag{15}$$

Using the relations (13)–(15), we can derive the terms $(P')^2, (Q')^2, P''$ and Q'' as follows:

$$(P')^2 = \frac{\Phi(\Psi)}{\Upsilon(\Psi)} \left(\sum_{i=0}^{\varsigma} i\tau_i \Psi^{i-1} \right)^2, \tag{16}$$

$$(Q')^2 = \frac{\Phi(\Psi)}{\Upsilon(\Psi)} \left(\sum_{i=0}^{\tilde{\varsigma}} i\tilde{\tau}_i \Psi^{i-1} \right)^2, \tag{17}$$

and

$$P'' = \frac{\Phi'(\Psi)\Upsilon(\Psi) - \Phi(\Psi)\Upsilon'(\Psi)}{2\Upsilon^2(\Psi)} \left(\sum_{i=0}^{\varsigma} i\tau_i \Psi^{i-1} \right) + \frac{\Phi(\Psi)}{\Upsilon(\Psi)} \left(\sum_{i=0}^{\varsigma} i(i-1)\tau_i \Psi^{i-2} \right), \tag{18}$$

$$Q'' = \frac{\Phi'(\Psi)\Upsilon(\Psi) - \Phi(\Psi)\Upsilon'(\Psi)}{2\Upsilon^2(\Psi)}\left(\sum_{i=0}^{\tilde{\varsigma}} i\tilde{\tau}_i\Psi^{i-1}\right) + \frac{\Phi(\Psi)}{\Upsilon(\Psi)}\left(\sum_{i=0}^{\tilde{\varsigma}} i(i-1)\tilde{\tau}_i\Psi^{i-2}\right), \tag{19}$$

where $\Phi(\Psi)$ and $\Upsilon(\Psi)$ are polynomials of Ψ. We can reduce Eq. (15) to the elementary integral form as follows:

$$\pm(\xi - \xi_0) = \int \frac{d\Psi}{\sqrt{\Lambda(\Psi)}} = \int \sqrt{\frac{\Upsilon(\Psi)}{\Phi(\Psi)}}d\Psi. \tag{20}$$

Using the balance principle, one finds that

$$\varsigma = \tilde{\varsigma} = \sigma - \rho - 2. \tag{21}$$

Case 1. When $\sigma = 3$, $\rho = 0$ and $\varsigma = \tilde{\varsigma} = 1$ in Eq. (21), we have

$$P = \tau_0 + \tau_1\Psi, \tag{22}$$

$$Q = \tilde{\tau}_0 + \tilde{\tau}_1\Psi, \tag{23}$$

$$P'' = \frac{\tau_1(3\mu_3\Psi^2 + 2\mu_2\Psi + \mu_1)}{2\chi_0}, \tag{24}$$

$$Q'' = \frac{\tilde{\tau}_1(3\mu_3\Psi^2 + 2\mu_2\Psi + \mu_1)}{2\chi_0}, \tag{25}$$

where $\mu_3 \neq 0$ and $\chi_0 \neq 0$. Substituting (22)–(25) into Eqs. (8) and (12), and solving the resulting system of algebraic equations, we obtain

$$\begin{aligned}
&\mu_0 = \mu_0, \quad \mu_1 = -\frac{2b\chi_0\tilde{\tau}_0(\alpha\tau_1^2\tilde{\tau}_0 + \lambda\tilde{\tau}_1^2)}{a\alpha\tau_1^2\tilde{\tau}_1 B^2},\\
&\mu_2 = -\frac{b\chi_0(2\alpha\tau_1^2\tilde{\tau}_0 + \lambda\tilde{\tau}_1^2)}{a\alpha\tau_1^2 B^2}, \quad \mu_3 = -\frac{2b\chi_0\tilde{\tau}_1}{3aB^2},\\
&\chi_0 = \chi_0, \quad \tau_1 = \tau_1, \quad \tilde{\tau}_0 = \tilde{\tau}_0, \quad \tilde{\tau}_1 = \tilde{\tau}_1,\\
&\tau_0 = \frac{\tau_1\tilde{\tau}_0}{\tilde{\tau}_1}, \quad c = \frac{a\alpha\tau_1^2}{b\tilde{\tau}_1^2}, \quad \omega = -a\kappa^2 - \frac{b\lambda\tilde{\tau}_1^2}{\alpha\tau_1^2}.
\end{aligned} \tag{26}$$

Substituting these results into Eqs. (15) and (20), we find that

$$\pm(\xi - \xi_0) = \sqrt{W_1}\int \frac{d\Psi}{\sqrt{\Lambda(\Psi)}}, \tag{27}$$

where

$$\Lambda(\Psi) = \Psi^3 + \frac{\mu_2}{\mu_3}\Psi^2 + \frac{\mu_1}{\mu_3}\Psi + \frac{\mu_0}{\mu_3}, \quad W_1 = \frac{\chi_0}{\mu_3}. \tag{28}$$

Consequently, one recovers the traveling wave solutions to (1) and (11) as follows:

When $\Lambda(\Psi) = (\Psi - \lambda_1)^3$,

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_1 + \frac{4\tau_1 W_1}{[B(x+2a\kappa t) - \xi_0]^2}\right\} \times \exp\left[i\left\{-\kappa x - \left(a\kappa^2 + \frac{b\lambda\tilde{\tau}_1^2}{\alpha\tau_1^2}\right)t + \theta\right\}\right], \tag{29}$$

$$r(x,t) = \left\{\tilde{\tau}_0 + \tilde{\tau}_1\lambda_1 + \frac{4\tilde{\tau}_1 W_1}{[B(x+2a\kappa t) - \xi_0]^2}\right\}. \tag{30}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)$ and $\lambda_2 > \lambda_1$,

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_2 + \tau_1(\lambda_1 - \lambda_2)\tanh^2\left(\frac{1}{2}\sqrt{\frac{\lambda_1 - \lambda_2}{W_1}}[B(x+2a\kappa t) - \xi_0]\right)\right\} \times \exp\left[i\left\{-\kappa x - \left(a\kappa^2 + \frac{b\lambda\tilde{\tau}_1^2}{\alpha\tau_1^2}\right)t + \theta\right\}\right], \tag{31}$$

$$r(x,t) = \left\{\tilde{\tau}_0 + \tilde{\tau}_1\lambda_2 + \tilde{\tau}_1(\lambda_1 - \lambda_2)\tanh^2\left(\frac{1}{2}\sqrt{\frac{\lambda_1 - \lambda_2}{W_1}}[B(x+2a\kappa t) - \xi_0]\right)\right\}. \tag{32}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)^2$ and $\lambda_1 > \lambda_2$,

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_1 + \tau_1(\lambda_1 - \lambda_2)\text{cosech}^2\left(\frac{B}{2}\sqrt{\frac{\lambda_1 - \lambda_2}{W_1}}(x+2a\kappa t)\right)\right\} \times \exp\left[i\left\{-\kappa x - \left(a\kappa^2 + \frac{b\lambda\tilde{\tau}_1^2}{\alpha\tau_1^2}\right)t + \theta\right\}\right], \tag{33}$$

$$r(x,t) = \left\{\tilde{\tau}_0 + \tilde{\tau}_1\lambda_1 + \tilde{\tau}_1(\lambda_1 - \lambda_2)\text{cosech}^2\left(\frac{B}{2}\sqrt{\frac{\lambda_1 - \lambda_2}{W_1}}(x+2a\kappa t)\right)\right\}. \tag{34}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)(\Psi - \lambda_3)$ and $\lambda_1 > \lambda_2 > \lambda_3$,

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_3 + \tau_1(\lambda_2 - \lambda_3)\text{sn}^2\left(\mp\frac{1}{2}\sqrt{\frac{\lambda_1 - \lambda_3}{W_1}}[B(x+2a\kappa t) - \xi_0], l\right)\right\} \times \exp\left[i\left\{-\kappa x - \left(a\kappa^2 + \frac{b\lambda\tilde{\tau}_1^2}{\alpha\tau_1^2}\right)t + \theta\right\}\right], \tag{35}$$

$$r(x,t) = \left\{\tilde{\tau}_0 + \tilde{\tau}_1\lambda_3 + \tilde{\tau}_1(\lambda_2 - \lambda_3)\text{sn}^2\left(\mp\frac{1}{2}\sqrt{\frac{\lambda_1 - \lambda_3}{W_1}}[B(x+2a\kappa t) - \xi_0], l\right)\right\}, \tag{36}$$

where

$$l^2 = \frac{\lambda_2 - \lambda_3}{\lambda_1 - \lambda_3}. \tag{37}$$

Note that λ_i $(i = 1, 2, 3)$ are the roots of the polynomial equation

$$\Lambda(\Psi) = 0. \tag{38}$$

When $\tau_0 = -\tau_1\lambda_1$, $\tilde{\tau}_0 = -\tilde{\tau}_1\lambda_1$ and $\xi_0 = 0$, the solutions (29)–(34) can be reduced to plane wave solutions

$$q(x,t) = \left\{\frac{A_1}{B(x+2a\kappa t)}\right\}^2 \exp\left[i\left\{-\kappa x - \left(a\kappa^2 + \frac{b\lambda\tilde{\tau}_1^2}{\alpha\tau_1^2}\right)t + \theta\right\}\right], \tag{39}$$

$$r(x,t) = \left\{\frac{\widetilde{A}_1}{B(x+2a\kappa t)}\right\}^2, \tag{40}$$

nematicon solutions

$$q(x,t) = \left\{\frac{A_2}{\cosh^2[B_1(x+2a\kappa t)]}\right\} \exp\left[i\left\{-\kappa x - \left(a\kappa^2 + \frac{b\lambda\tilde{\tau}_1^2}{\alpha\tau_1^2}\right)t + \theta\right\}\right], \tag{41}$$

$$r(x,t) = \left\{\frac{\widetilde{A}_2}{\cosh^2[B_1(x+2a\kappa t)]}\right\}, \tag{42}$$

and singular soliton solutions

$$q(x,t) = \left\{\frac{A_3}{\sinh^2[B_1(x+2a\kappa t)]}\right\} \exp\left[i\left\{-\kappa x - \left(a\kappa^2 + \frac{b\lambda\tilde{\tau}_1^2}{\alpha\tau_1^2}\right)t + \theta\right\}\right], \tag{43}$$

$$r(x,t) = \left\{\frac{\widetilde{A}_3}{\sinh^2[B_1(x+2a\kappa t)]}\right\}, \tag{44}$$

where

$$\begin{aligned} &A_1 = 2\sqrt{\tau_1 W_1}, \quad A_2 = \tau_1(\lambda_2 - \lambda_1), \quad A_3 = \tau_1(\lambda_1 - \lambda_2),\\ &\widetilde{A}_1 = 2\sqrt{\tilde{\tau}_1 W_1}, \quad \widetilde{A}_2 = \tilde{\tau}_1(\lambda_2 - \lambda_1),\\ &\widetilde{A}_3 = \tilde{\tau}_1(\lambda_1 - \lambda_2), \quad B_1 = \frac{B}{2}\sqrt{\frac{\lambda_1 - \lambda_2}{W_1}}. \end{aligned} \tag{45}$$

Here, $A_2, \widetilde{A}_2, A_3$ and $\widetilde{A}_3$ are, respectively, the amplitudes of 1-soliton and singular soliton, while B_1 is the inverse width of the solitons. These solitons are valid for $\tau_1 > 0$ and $\tilde{\tau}_1 > 0$. Moreover, when $\tau_0 = -\tau_1\lambda_3$, $\tilde{\tau}_0 = -\tilde{\tau}_1\lambda_3$ and $\xi_0 = 0$, Jacobi elliptic function solutions (35) and (36) can be simplified as

$$q(x,t) = A_4\mathrm{sn}^2\left[B_j(x+2a\kappa t), \frac{\lambda_2-\lambda_3}{\lambda_1-\lambda_3}\right] \exp\left[i\left\{-\kappa x - \left(a\kappa^2 + \frac{b\lambda\tilde{\tau}_1^2}{\alpha\tau_1^2}\right)t + \theta\right\}\right], \tag{46}$$

$$r(x,t) = \widetilde{A}_4\mathrm{sn}^2\left[B_j(x+2a\kappa t), \frac{\lambda_2-\lambda_3}{\lambda_1-\lambda_3}\right], \tag{47}$$

where

$$A_4 = \tau_1(\lambda_2 - \lambda_3), \quad \widetilde{A}_4 = \tilde{\tau}_1(\lambda_2 - \lambda_3), \quad B_j = \frac{(-1)^j B}{2}\sqrt{\frac{\lambda_1 - \lambda_3}{W_1}}, \ \ (j = 2, 3). \tag{48}$$

Remark 1. When the modulus $l \to 1$, the following shock wave solutions emerge:

$$q(x,t) = A_4 \tanh^2[B_j(x + 2a\kappa t)] \exp\left[i\left\{-\kappa x - \left(a\kappa^2 + \frac{b\lambda\tilde{\tau}_1^2}{\alpha\tau_1^2}\right)t + \theta\right\}\right], \tag{49}$$

$$r(x,t) = \widetilde{A}_4 \tanh^2[B_j(x + 2a\kappa t)], \tag{50}$$

where $\lambda_1 = \lambda_2$.

Case 2. When $\sigma = 4$, $\rho = 0$ and $\varsigma = \tilde{\varsigma} = 2$ in Eq. (21), one recovers that

$$P = \tau_0 + \tau_1\Psi + \tau_2\Psi^2, \tag{51}$$

$$Q = \tilde{\tau}_0 + \tilde{\tau}_1\Psi + \tilde{\tau}_2\Psi^2, \tag{52}$$

$$(P')^2 = \frac{(\tau_1 + 2\tau_2\Psi)^2(\mu_4\Psi^4 + \mu_3\Psi^3 + \mu_2\Psi^2 + \mu_1\Psi + \mu_0)}{\chi_0}, \tag{53}$$

$$(Q')^2 = \frac{(\tilde{\tau}_1 + 2\tilde{\tau}_2\Psi)^2(\mu_4\Psi^4 + \mu_3\Psi^3 + \mu_2\Psi^2 + \mu_1\Psi + \mu_0)}{\chi_0}, \tag{54}$$

$$P'' = \frac{\begin{array}{l}(\tau_1 + 2\tau_2\Psi)(4\mu_4\Psi^3 + 3\mu_3\Psi^2 + 2\mu_2\Psi + \mu_1)\\ \quad + 4\tau_2(\mu_4\Psi^4 + \mu_3\Psi^3 + \mu_2\Psi^2 + \mu_1\Psi + \mu_0)\end{array}}{2\chi_0}, \tag{55}$$

$$Q'' = \frac{\begin{array}{l}(\tilde{\tau}_1 + 2\tilde{\tau}_2\Psi)(4\mu_4\Psi^3 + 3\mu_3\Psi^2 + 2\mu_2\Psi + \mu_1)\\ \quad + 4\tilde{\tau}_2(\mu_4\Psi^4 + \mu_3\Psi^3 + \mu_2\Psi^2 + \mu_1\Psi + \mu_0)\end{array}}{2\chi_0}, \tag{56}$$

where $\mu_4 \neq 0$ and $\chi_0 \neq 0$. Substituting (51)–(56) into Eqs. (8) and (12), and solving the resulting system of algebraic equations, we have the following results:

$$\mu_1 = -\frac{4\mu_0\tau_1\tau_2\tilde{\tau}_2[bc\tau_1^2\tilde{\tau}_2 - 6\tau_2^2(a\lambda + 2bc\tilde{\tau}_0)]}{bc\tau_1^4\tilde{\tau}_2^2 - 6\tau_1^2\tau_2^2\tilde{\tau}_2(a\lambda + 2bc\tilde{\tau}_0) + 48\tau_2^4\tilde{\tau}_0(a\lambda + bc\tilde{\tau}_0)},$$

$$\mu_2 = \frac{12\mu_0\tau_2^2\tilde{\tau}_2[bc\tau_1^2\tilde{\tau}_2 + 2\tau_2^2(a\lambda + 2bc\tilde{\tau}_0)]}{bc\tau_1^4\tilde{\tau}_2^2 - 6\tau_1^2\tau_2^2\tilde{\tau}_2(a\lambda + 2bc\tilde{\tau}_0) + 48\tau_2^4\tilde{\tau}_0(a\lambda + bc\tilde{\tau}_0)},$$

$$\mu_3 = \frac{32bc\mu_0\tau_1\tau_2^3\tilde{\tau}_2^2}{bc\tau_1^4\tilde{\tau}_2^2 - 6\tau_1^2\tau_2^2\tilde{\tau}_2(a\lambda + 2bc\tilde{\tau}_0) + 48\tau_2^4\tilde{\tau}_0(a\lambda + bc\tilde{\tau}_0)},$$

$$\mu_4 = \frac{16bc\mu_0\tau_2^4\tilde{\tau}_2^2}{bc\tau_1^4\tilde{\tau}_2^2 - 6\tau_1^2\tau_2^2\tilde{\tau}_2(a\lambda + 2bc\tilde{\tau}_0) + 48\tau_2^4\tilde{\tau}_0(a\lambda + bc\tilde{\tau}_0)},$$

$$\chi_0 = -\frac{96ac\mu_0\tau_2^4\tilde{\tau}_2B^2}{bc\tau_1^4\tilde{\tau}_2^2 - 6\tau_1^2\tau_2^2\tilde{\tau}_2(a\lambda + 2bc\tilde{\tau}_0) + 48\tau_2^4\tilde{\tau}_0(a\lambda + bc\tilde{\tau}_0)},$$

$$\alpha = \frac{bc\tilde{\tau}_2^2}{a\tau_2^2}, \quad \tau_0 = \frac{\tau_2\tilde{\tau}_0}{\tilde{\tau}_2}, \quad \tilde{\tau}_1 = \frac{\tau_1\tilde{\tau}_2}{\tau_2}, \quad \omega = -\frac{a(c\kappa^2 + \lambda)}{c},$$

$$\mu_0 = \mu_0, \quad \tau_1 = \tau_1, \quad \tau_2 = \tau_2, \quad \tilde{\tau}_0 = \tilde{\tau}_0, \quad \tilde{\tau}_2 = \tilde{\tau}_2. \tag{57}$$

Substituting these results into Eqs. (15) and (20), we find that

$$\pm(\xi - \xi_0) = W_2 \int \frac{d\Psi}{\sqrt{\Lambda(\Psi)}}, \tag{58}$$

where

$$\Lambda(\Psi) = \Psi^4 + \frac{\mu_3}{\mu_4}\Psi^3 + \frac{\mu_2}{\mu_4}\Psi^2 + \frac{\mu_1}{\mu_4}\Psi + \frac{\mu_0}{\mu_4}, \quad W_2 = \sqrt{\frac{\chi_0}{\mu_4}}. \tag{59}$$

Integrating Eq. (58) and taking $\xi_0 = 0$, one obtains the traveling wave solutions to Eqs. (1) and (11) in the forms:

When $\Lambda(\Psi) = (\Psi - \lambda_1)^4$,

$$q(x,t) = \sum_{j=0}^{2} \tau_j \left(\lambda_1 \pm \frac{W_2}{B(x + 2a\kappa t)}\right)^j \exp\left[i\left\{-\kappa x - \frac{a(c\kappa^2 + \lambda)}{c}t + \theta\right\}\right], \tag{60}$$

$$r(x,t) = \sum_{j=0}^{2} \tilde{\tau}_j \left(\lambda_1 \pm \frac{W_2}{B(x + 2a\kappa t)}\right)^j. \tag{61}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^3(\Psi - \lambda_2)$ and $\lambda_2 > \lambda_1$,

$$q(x,t) = \sum_{j=0}^{2} \tau_j \left(\lambda_1 + \frac{4W_2^2(\lambda_2 - \lambda_1)}{4W_2^2 - [B(\lambda_1 - \lambda_2)(x + 2a\kappa t)]^2}\right)^j$$

$$\times \exp\left[i\left\{-\kappa x - \frac{a(c\kappa^2 + \lambda)}{c}t + \theta\right\}\right], \tag{62}$$

$$r(x,t) = \sum_{j=0}^{2} \tilde{\tau}_j \left(\lambda_1 + \frac{4W_2^2(\lambda_2 - \lambda_1)}{4W_2^2 - [B(\lambda_1 - \lambda_2)(x + 2a\kappa t)]^2}\right)^j. \tag{63}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)^2$,

$$q(x,t) = \sum_{j=0}^{2} \tau_j \left(\lambda_2 + \frac{\lambda_2 - \lambda_1}{\exp\left[\frac{B(\lambda_1 - \lambda_2)}{W_2}(x + 2a\kappa t)\right] - 1}\right)^j$$

$$\times \exp\left[i\left\{-\kappa x-\frac{a\left(c\kappa^2+\lambda\right)}{c}t+\theta\right\}\right], \tag{64}$$

$$r(x,t)=\sum_{j=0}^{2}\tilde{\tau}_j\left(\lambda_2+\frac{\lambda_2-\lambda_1}{\exp\left[\frac{B(\lambda_1-\lambda_2)}{W_2}(x+2a\kappa t)\right]-1}\right)^j \tag{65}$$

and

$$q(x,t)=\sum_{j=0}^{2}\tau_j\left(\lambda_1+\frac{\lambda_1-\lambda_2}{\exp\left[\frac{B(\lambda_1-\lambda_2)}{W_2}(x+2a\kappa t)\right]-1}\right)^j$$

$$\times \exp\left[i\left\{-\kappa x-\frac{a\left(c\kappa^2+\lambda\right)}{c}t+\theta\right\}\right], \tag{66}$$

$$r(x,t)=\sum_{j=0}^{2}\tilde{\tau}_j\left(\lambda_1+\frac{\lambda_1-\lambda_2}{\exp\left[\frac{B(\lambda_1-\lambda_2)}{W_2}(x+2a\kappa t)\right]-1}\right)^j. \tag{67}$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)^2(\Psi-\lambda_2)(\Psi-\lambda_3)$ and $\lambda_1>\lambda_2>\lambda_3$,

$$q(x,t)=\sum_{j=0}^{2}\tau_j\left(\lambda_1-\frac{2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{2\lambda_1-\lambda_2-\lambda_3+(\lambda_3-\lambda_2)\times\cosh\left(\frac{B\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{W_2}(x+2a\kappa t)\right)}\right)^j$$

$$\times \exp\left[i\left\{-\kappa x-\frac{a(c\kappa^2+\lambda)}{c}t+\theta\right\}\right], \tag{68}$$

$$r(x,t)=\sum_{j=0}^{2}\tilde{\tau}_j\left(\lambda_1-\frac{2(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{2\lambda_1-\lambda_2-\lambda_3+(\lambda_3-\lambda_2)\times\cosh\left(\frac{B\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{W_2}(x+2a\kappa t)\right)}\right)^j. \tag{69}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)(\Psi - \lambda_3)(\Psi - \lambda_4)$ and $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$,

$$q(x,t) = \sum_{j=0}^{2} \tau_j \left(\lambda_2 + \frac{(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_4 - \lambda_2 + (\lambda_1 - \lambda_4)\mathrm{sn}^2 \times \left[\pm \frac{B\sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}}{2W_2}(x + 2a\kappa t), l \right]} \right)^j \times \exp\left[i \left\{ -\kappa x - \frac{a(c\kappa^2 + \lambda)}{c} t + \theta \right\} \right], \tag{70}$$

$$r(x,t) = \sum_{j=0}^{2} \tilde{\tau}_j \left(\lambda_2 + \frac{(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_4 - \lambda_2 + (\lambda_1 - \lambda_4)\mathrm{sn}^2 \times \left[\pm \frac{B\sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}}{2W_2}(x + 2a\kappa t), l \right]} \right)^j, \tag{71}$$

where

$$l^2 = \frac{(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_4)}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}. \tag{72}$$

Note that λ_i $(i = 1, \ldots, 4)$ are the roots of the polynomial equation

$$\Lambda(\Psi) = 0. \tag{73}$$

Remark 2. When the modulus $l \to 1$, singular soliton solutions fall out

$$q(x,t) = \sum_{j=0}^{2} \tau_j \left(\lambda_2 + \frac{(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_4 - \lambda_2 + (\lambda_1 - \lambda_4)\tanh^2 \times \left[\pm \frac{B\sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}}{2W_2}(x + 2a\kappa t) \right]} \right)^j \times \exp\left[i \left\{ -\kappa x - \frac{a(c\kappa^2 + \lambda)}{c} t + \theta \right\} \right], \tag{74}$$

$$r(x,t) = \sum_{j=0}^{2} \tilde{\tau}_j \left(\lambda_2 + \frac{(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_4 - \lambda_2 + (\lambda_1 - \lambda_4)\tanh^2 \times \left[\pm \frac{B\sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}}{2W_2}(x + 2a\kappa t) \right]} \right)^j, \tag{75}$$

where $\lambda_3 = \lambda_4$.

Remark 3. However, if $l \to 0$, the following periodic singular waves are listed:

$$q(x,t) = \sum_{j=0}^{2} \tau_j \left(\lambda_2 + \frac{(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\sin^2 \times \left[\pm \frac{B\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W_2}(x+2a\kappa t)\right]} \right)^j \times \exp\left[i\left\{-\kappa x - \frac{a(c\kappa^2+\lambda)}{c}t+\theta\right\}\right], \tag{76}$$

$$r(x,t) = \sum_{j=0}^{2} \tilde{\tau}_j \left(\lambda_2 + \frac{(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\sin^2 \times \left[\pm \frac{B\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W_2}(x+2a\kappa t)\right]} \right)^j, \tag{77}$$

where $\lambda_2 = \lambda_3$.

3. Power Law

For power law nonlinearity,

$$F(s) = s^n. \tag{78}$$

In this case, Eq. (2) reduces to

$$cr_{xx} + \lambda r + \alpha |q|^{2n} = 0. \tag{79}$$

In this case, Eq. (9) simplifies to

$$cB^2 Q'' + \lambda Q + \alpha P^{2n} = 0. \tag{80}$$

In order to obtain closed form solutions, we use the transformation

$$P = U_1^{\frac{2}{n}} \quad \text{and} \quad Q = U_2^2, \tag{81}$$

that will reduce Eqs. (8) and (80) into the ordinary differential equations (ODEs)

$$-(\omega + a\kappa^2)U_1^2 + aB^2 \left\{ \frac{2(2-n)}{n^2}(U_1')^2 + \frac{2}{n}U_1 U_1'' \right\} + bU_1^2 U_2^2 = 0, \tag{82}$$

and

$$\alpha U_1^4 + \lambda U_2^2 + 2cB^2\{(U_2')^2 + U_2 U_2''\} = 0. \tag{83}$$

By extended trial equation method, Eqs. (82) and (83) will now be analyzed in the remaining section for soliton and other solutions. Using the balance principle, one finds that

$$\varsigma = \tilde{\varsigma} = \frac{\sigma - \rho - 2}{2}. \tag{84}$$

When $\sigma = 4$, $\rho = 0$ and $\varsigma = \tilde{\varsigma} = 1$ in Eq. (84), we have

$$U_1 = \tau_0 + \tau_1 \Psi, \tag{85}$$

$$U_2 = \tilde{\tau}_0 + \tilde{\tau}_1 \Psi, \tag{86}$$

$$(U_1')^2 = \frac{\tau_1^2(\mu_4\Psi^4 + \mu_3\Psi^3 + \mu_2\Psi^2 + \mu_1\Psi + \mu_0)}{\chi_0}, \tag{87}$$

$$(U_2')^2 = \frac{\tilde{\tau}_1^2(\mu_4\Psi^4 + \mu_3\Psi^3 + \mu_2\Psi^2 + \mu_1\Psi + \mu_0)}{\chi_0}, \tag{88}$$

$$U_1'' = \frac{\tau_1(4\mu_4\Psi^3 + 3\mu_3\Psi^2 + 2\mu_2\Psi + \mu_1)}{2\chi_0}, \tag{89}$$

$$U_2'' = \frac{\tilde{\tau}_1(4\mu_4\Psi^3 + 3\mu_3\Psi^2 + 2\mu_2\Psi + \mu_1)}{2\chi_0}, \tag{90}$$

where $\mu_4 \neq 0$ and $\chi_0 \neq 0$. Substituting (85)–(90) into Eqs. (82) and (83), collecting the coefficients of Ψ, and solving the resulting algebraic equations system, we have the following results:

$$\begin{aligned}
&\mu_0 = -\frac{\chi_0\tilde{\tau}_0^2(2\alpha\tau_1^4\tilde{\tau}_0^2 + 3\lambda\tilde{\tau}_1^4)}{12c\tilde{\tau}_1^6 B^2}, \quad \mu_3 = -\frac{2\alpha\tau_1^4\chi_0\tilde{\tau}_0}{3c\tilde{\tau}_1^3 B^2}, \quad \tau_0 = \frac{\tau_1\tilde{\tau}_0}{\tilde{\tau}_1}, \\
&\mu_1 = -\frac{\chi_0\tilde{\tau}_0(4\alpha\tau_1^4\tilde{\tau}_0^2 + 3\lambda\tilde{\tau}_1^4)}{6c\tilde{\tau}_1^5 B^2}, \quad \mu_4 = -\frac{\alpha\tau_1^4\chi_0}{6c\tilde{\tau}_1^2 B^2}, \quad \omega = -\frac{3b\tilde{\tau}_1^4(cn^2\kappa^2 + \lambda)}{\alpha\tau_1^4(2+n)}, \\
&\mu_2 = -\frac{\chi_0(4\alpha\tau_1^4\tilde{\tau}_0^2 + \lambda\tilde{\tau}_1^4)}{4c\tilde{\tau}_1^4 B^2}, \quad a = \frac{3bcn^2\tilde{\tau}_1^4}{\alpha\tau_1^4(2+n)}, \\
&\chi_0 = \chi_0, \quad \tau_1 = \tau_1, \quad \tilde{\tau}_0 = \tilde{\tau}_0, \quad \tilde{\tau}_1 = \tilde{\tau}_1.
\end{aligned} \tag{91}$$

Substituting these results into Eqs. (15) and (20), we find that

$$\pm(\xi - \xi_0) = W_3 \int \frac{d\Psi}{\sqrt{\Lambda(\Psi)}}, \tag{92}$$

where

$$\Lambda(\Psi) = \Psi^4 + \frac{\mu_3}{\mu_4}\Psi^3 + \frac{\mu_2}{\mu_4}\Psi^2 + \frac{\mu_1}{\mu_4}\Psi + \frac{\mu_0}{\mu_4}, \quad W_3 = \sqrt{\frac{\chi_0}{\mu_4}}. \tag{93}$$

Consequently, one recovers the traveling wave solutions to (1) and (79) as follows:

When $\Lambda(\Psi) = (\Psi - \lambda_1)^4$,

$$\begin{aligned}
q(x,t) = &\left\{\tau_0 + \tau_1\lambda_1 \pm \frac{\tau_1 W_3}{B(x + 2a\kappa t) - \xi_0}\right\}^{\frac{2}{n}} \\
&\times \exp\left[i\left\{-\kappa x - \frac{3b\tilde{\tau}_1^4(cn^2\kappa^2 + \lambda)}{\alpha\tau_1^4(2+n)}t + \theta\right\}\right],
\end{aligned} \tag{94}$$

$$r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_1\pm\frac{\tilde{\tau}_1 W_3}{B(x+2a\kappa t)-\xi_0}\right\}^2. \tag{95}$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)^3(\Psi-\lambda_2)$ and $\lambda_2>\lambda_1$,

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_1+\frac{4\tau_1 W_3^2(\lambda_2-\lambda_1)}{4W_3^2-[(\lambda_1-\lambda_2)(B(x+2a\kappa t)-\xi_0)]^2}\right\}^{\frac{2}{n}}$$

$$\times\exp\left[i\left\{-\kappa x-\frac{3b\tilde{\tau}_1^4(cn^2\kappa^2+\lambda)}{\alpha\tau_1^4(2+n)}t+\theta\right\}\right], \tag{96}$$

$$r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_1+\frac{4\tilde{\tau}_1 W_3^2(\lambda_2-\lambda_1)}{4W_3^2-[(\lambda_1-\lambda_2)(B(x+2a\kappa t)-\xi_0)]^2}\right\}^2. \tag{97}$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)^2(\Psi-\lambda_2)^2$,

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_2+\frac{\tau_1(\lambda_2-\lambda_1)}{\exp\left[\frac{\lambda_1-\lambda_2}{W_3}(B(x+2a\kappa t)-\xi_0)\right]-1}\right\}^{\frac{2}{n}}$$

$$\times\exp\left[i\left\{-\kappa x-\frac{3b\tilde{\tau}_1^4(cn^2\kappa^2+\lambda)}{\alpha\tau_1^4(2+n)}t+\theta\right\}\right], \tag{98}$$

$$r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_2+\frac{\tilde{\tau}_1(\lambda_2-\lambda_1)}{\exp\left[\frac{\lambda_1-\lambda_2}{W_3}(B(x+2a\kappa t)-\xi_0)\right]-1}\right\}^2, \tag{99}$$

and

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_1+\frac{\tau_1(\lambda_1-\lambda_2)}{\exp\left[\frac{\lambda_1-\lambda_2}{W_3}(B(x+2a\kappa t)-\xi_0)\right]-1}\right\}^{\frac{2}{n}}$$

$$\times\exp\left[i\left\{-\kappa x-\frac{3b\tilde{\tau}_1^4(cn^2\kappa^2+\lambda)}{\alpha\tau_1^4(2+n)}t+\theta\right\}\right], \tag{100}$$

$$r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_1+\frac{\tilde{\tau}_1(\lambda_1-\lambda_2)}{\exp\left[\frac{\lambda_1-\lambda_2}{W_3}(B(x+2a\kappa t)-\xi_0)\right]-1}\right\}^2. \tag{101}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)(\Psi - \lambda_3)$ and $\lambda_1 > \lambda_2 > \lambda_3$,

$$q(x,t) = \left\{ \tau_0 + \tau_1\lambda_1 - \frac{2\tau_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}{2\lambda_1 - \lambda_2 - \lambda_3 + (\lambda_3 - \lambda_2)\cosh \times \left[\frac{B\sqrt{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}}{W_3}(x + 2a\kappa t) \right]} \right\}^{\frac{2}{n}}$$
$$\times \exp\left[i\left\{ -\kappa x - \frac{3b\tilde{\tau}_1^4(cn^2\kappa^2 + \lambda)}{\alpha\tau_1^4(2+n)}t + \theta \right\} \right], \tag{102}$$

$$r(x,t) = \left\{ \tilde{\tau}_0 + \tilde{\tau}_1\lambda_1 - \frac{2\tilde{\tau}_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}{2\lambda_1 - \lambda_2 - \lambda_3 + (\lambda_3 - \lambda_2)\cosh \times \left[\frac{B\sqrt{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}}{W_3}(x + 2a\kappa t) \right]} \right\}^{2}. \tag{103}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)(\Psi - \lambda_2)(\Psi - \lambda_3)(\Psi - \lambda_4)$ and $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4$,

$$q(x,t) = \left\{ \tau_0 + \tau_1\lambda_2 + \frac{\tau_1(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_4 - \lambda_2 + (\lambda_1 - \lambda_4)\mathrm{sn}^2 \times \left[\pm\frac{\sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}}{2W_3}(B(x + 2a\kappa t) - \xi_0), l \right]} \right\}^{\frac{2}{n}}$$
$$\times \exp\left[i\left\{ -\kappa x - \frac{3b\tilde{\tau}_1^4(cn^2\kappa^2 + \lambda)}{\alpha\tau_1^4(2+n)}t + \theta \right\} \right], \tag{104}$$

$$r(x,t) = \left\{ \tilde{\tau}_0 + \tilde{\tau}_1\lambda_2 + \frac{\tilde{\tau}_1(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_4 - \lambda_2 + (\lambda_1 - \lambda_4)\mathrm{sn}^2 \times \left[\pm\frac{\sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}}{2W_3}(B(x + 2a\kappa t) - \xi_0), l \right]} \right\}^{2}, \tag{105}$$

where

$$l^2 = \frac{(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_4)}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}. \tag{106}$$

Note that λ_i $(i = 1, \ldots, 4)$ are the roots of the polynomial equation

$$\Lambda(\Psi) = 0. \tag{107}$$

When $\tau_0 = -\tau_1\lambda_1$, $\tilde{\tau}_0 = -\tilde{\tau}_1\lambda_1$ and $\xi_0 = 0$, the solutions (94)–(103) can be reduced to plane wave solutions

$$q(x,t) = \left\{\pm\frac{\tau_1 W_3}{B(x+2a\kappa t)}\right\}^{\frac{2}{n}} \exp\left[i\left\{-\kappa x - \frac{3b\tilde{\tau}_1^4(cn^2\kappa^2+\lambda)}{\alpha\tau_1^4(2+n)}t + \theta\right\}\right], \tag{108}$$

$$r(x,t) = \left\{\pm\frac{\tilde{\tau}_1 W_3}{B(x+2a\kappa t)}\right\}^{2}, \tag{109}$$

$$q(x,t) = \left\{\frac{4\tau_1 W_3^2(\lambda_2-\lambda_1)}{4W_3^2 - [B(\lambda_1-\lambda_2)(x+2a\kappa t)]^2}\right\}^{\frac{2}{n}} \times \exp\left[i\left\{-\kappa x - \frac{3b\tilde{\tau}_1^4(cn^2\kappa^2+\lambda)}{\alpha\tau_1^4(2+n)}t + \theta\right\}\right], \tag{110}$$

$$r(x,t) = \left\{\frac{4\tilde{\tau}_1 W_3^2(\lambda_2-\lambda_1)}{4W_3^2 - [B(\lambda_1-\lambda_2)(x+2a\kappa t)]^2}\right\}^{2}, \tag{111}$$

singular soliton solutions

$$q(x,t) = \left\{\frac{\tau_1(\lambda_2-\lambda_1)}{2}\left(1 \mp \coth\left[\frac{B(\lambda_1-\lambda_2)}{2W_3}(x+2a\kappa t)\right]\right)\right\}^{\frac{2}{n}} \times \exp\left[i\left\{-\kappa x - \frac{3b\tilde{\tau}_1^4(cn^2\kappa^2+\lambda)}{\alpha\tau_1^4(2+n)}t + \theta\right\}\right], \tag{112}$$

$$r(x,t) = \left\{\frac{\tilde{\tau}_1(\lambda_2-\lambda_1)}{2}\left(1 \mp \coth\left[\frac{B(\lambda_1-\lambda_2)}{2W_3}(x+2a\kappa t)\right]\right)\right\}^{2}, \tag{113}$$

and nematicon solutions

$$q(x,t) = \left\{\frac{A_5}{(C_1+\cosh[B_4(x+2a\kappa t)])^{\frac{2}{n}}}\right\} \times \exp\left[i\left\{-\kappa x - \frac{3b\tilde{\tau}_1^4(cn^2\kappa^2+\lambda)}{\alpha\tau_1^4(2+n)}t + \theta\right\}\right], \tag{114}$$

$$r(x,t) = \left\{\frac{\widetilde{A}_5}{(C_1+\cosh[B_4(x+2a\kappa t)])^{2}}\right\}, \tag{115}$$

where

$$A_5 = \left(\frac{2\tau_1(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{\lambda_3-\lambda_2}\right)^{\frac{2}{n}}, \quad \widetilde{A}_5 = \left(\frac{2\tilde{\tau}_1(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{\lambda_3-\lambda_2}\right)^{2}, \quad B_4 = \frac{B\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{W_3}, \quad C_1 = \frac{2\lambda_1-\lambda_2-\lambda_3}{\lambda_3-\lambda_2}. \tag{116}$$

Here, A_5 and $\widetilde{A}_5$ are the amplitudes of the solitons, while B_4 is the inverse width of the solitons. These solitons are valid for $\tau_1 < 0$ and $\tilde{\tau}_1 < 0$. Moreover, when $\tau_0 = -\tau_1\lambda_2$ and $\xi_0 = 0$, Jacobi elliptic function solutions (104) and (105) can be reduced as follows:

$$q(x,t) = \left\{ \frac{A_6}{\left(C_2 + \mathrm{sn}^2\left[B_j(x+2a\kappa t), \frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}\right]\right)^{\frac{2}{n}}} \right\} \times \exp\left[i\left\{-\kappa x - \frac{3b\tilde{\tau}_1^4(cn^2\kappa^2+\lambda)}{\alpha\tau_1^4(2+n)}t + \theta\right\}\right], \tag{117}$$

$$r(x,t) = \left\{ \frac{\widetilde{A}_6}{\left(C_2 + \mathrm{sn}^2\left[B_j(x+2a\kappa t), \frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}\right]\right)^{2}} \right\}, \tag{118}$$

where

$$A_6 = \left(\frac{\tau_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_1-\lambda_4}\right)^{\frac{2}{n}}, \quad \widetilde{A}_6 = \left(\frac{\tilde{\tau}_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_1-\lambda_4}\right)^{2},$$
$$C_2 = \frac{\lambda_4-\lambda_2}{\lambda_1-\lambda_4}, \quad B_j = \frac{(-1)^j B\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W_3}, \quad (j=5,6). \tag{119}$$

Remark 4. When the modulus $l \to 1$, the following singular soliton solutions emerge:

$$q(x,t) = \left\{ \frac{A_6}{(C_2 + \tanh^2[B_j(x+2a\kappa t)])^{\frac{2}{n}}} \right\} \times \exp\left[i\left\{-\kappa x - \frac{3b\tilde{\tau}_1^4(cn^2\kappa^2+\lambda)}{\alpha\tau_1^4(2+n)}t + \theta\right\}\right], \tag{120}$$

$$r(x,t) = \left\{ \frac{\widetilde{A}_6}{(C_2 + \tanh^2[B_j(x+2a\kappa t)])^{2}} \right\}, \tag{121}$$

where $\lambda_3 = \lambda_4$.

Remark 5. However, if $l \to 0$, singular periodic wave solutions are listed as

$$q(x,t) = \left\{ \frac{A_6}{(C_2 + \sin^2[B_j(x+2a\kappa t)])^{\frac{2}{n}}} \right\} \times \exp\left[i\left\{-\kappa x - \frac{3b\tilde{\tau}_1^4(cn^2\kappa^2+\lambda)}{\alpha\tau_1^4(2+n)}t + \theta\right\}\right], \tag{122}$$

$$r(x,t)=\left\{\frac{\widetilde{A}_6}{(C_2+\sin^2[B_j(x+2a\kappa t)])^2}\right\}, \tag{123}$$

where $\lambda_2=\lambda_3$.

4. Parabolic Law

For parabolic law set, the nonlinearity follows the form:

$$F(s)=c_1s+c_2s^2, \tag{124}$$

where c_l for $l=1,2$ are constants. Thus, Eq. (2) reduces to

$$cr_{xx}+\lambda r+\alpha(c_1|q|^2+c_2|q|^4)=0. \tag{125}$$

In this case, Eq. (9) simplifies to

$$cB^2Q''+\lambda Q+\alpha(c_1P^2+c_2P^4)=0. \tag{126}$$

In order to obtain closed form solutions, we use the transformation

$$P=U_1 \quad \text{and} \quad Q=U_2^2 \tag{127}$$

that will reduce Eqs. (8) and (126) into the ODEs

$$-(\omega+a\kappa^2)U_1+aB^2U_1''+bU_1U_2^2=0, \tag{128}$$

and

$$\alpha(c_1U_1^2+c_2U_1^4)+\lambda U_2^2+2cB^2\{(U_2')^2+U_2U_2''\}=0. \tag{129}$$

By extended trial equation method, Eqs. (128) and (129) will now be analyzed in the remaining section for soliton and other solutions. Using the balance principle, one finds that

$$\varsigma=\tilde{\varsigma}=\frac{\sigma-\rho-2}{2}. \tag{130}$$

Let us take $\sigma=4$, $\rho=0$ and $\varsigma=\tilde{\varsigma}=1$ in Eq. (130). Then, substituting the terms in Eqs. (13)–(19) into Eqs. (128) and (129), and solving the resulting system of algebraic equations, we obtain

$$\mu_0=-\frac{\chi_0\tilde{\tau}_0^2(2c_2\alpha\tau_1^4\tilde{\tau}_0^2+3c_1\alpha\tau_1^2\tilde{\tau}_1^2+3\lambda\tilde{\tau}_1^4)}{12c\tilde{\tau}_1^6B^2},$$

$$\mu_1=-\frac{\chi_0\tilde{\tau}_0(4c_2\alpha\tau_1^4\tilde{\tau}_0^2+3c_1\alpha\tau_1^2\tilde{\tau}_1^2+3\lambda\tilde{\tau}_1^4)}{6c\tilde{\tau}_1^5B^2},$$

$$\mu_2=-\frac{\chi_0(4c_2\alpha\tau_1^4\tilde{\tau}_0^2+c_1\alpha\tau_1^2\tilde{\tau}_1^2+\lambda\tilde{\tau}_1^4)}{4c\tilde{\tau}_1^4B^2},$$

$$\mu_3=-\frac{2c_2\alpha\tau_1^4\chi_0\tilde{\tau}_0}{3c\tilde{\tau}_1^3B^2}, \quad \mu_4=-\frac{c_2\alpha\tau_1^4\chi_0}{6c\tilde{\tau}_1^2B^2},$$

$$a=\frac{3bc\tilde{\tau}_1^4}{c_2\alpha\tau_1^4}, \quad \tau_0=\frac{\tau_1\tilde{\tau}_0}{\tilde{\tau}_1},$$

$$\omega = -\frac{3b\tilde{\tau}_1^2[c_1\alpha\tau_1^2 + \tilde{\tau}_1^2(4c\kappa^2 + \lambda)]}{4c_2\alpha\tau_1^4},$$
$$\chi_0 = \chi_0, \quad \tau_1 = \tau_1, \quad \tilde{\tau}_0 = \tilde{\tau}_0, \quad \tilde{\tau}_1 = \tilde{\tau}_1. \tag{131}$$

Substituting these results into Eqs. (15) and (20), we find that

$$\pm(\xi - \xi_0) = W_4 \int \frac{d\Psi}{\sqrt{\Lambda(\Psi)}}, \tag{132}$$

where

$$\Lambda(\Psi) = \Psi^4 + \frac{\mu_3}{\mu_4}\Psi^3 + \frac{\mu_2}{\mu_4}\Psi^2 + \frac{\mu_1}{\mu_4}\Psi + \frac{\mu_0}{\mu_4}, \quad W_4 = \sqrt{\frac{\chi_0}{\mu_4}}. \tag{133}$$

Consequently, one recovers the traveling wave solutions to (1) and (125) as follows: When $\Lambda(\Psi) = (\Psi - \lambda_1)^4$,

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_1 \pm \frac{\tau_1 W_4}{B(x + 2a\kappa t) - \xi_0}\right\} \times \exp\left[i\left\{-\kappa x - \left(\frac{3b\tilde{\tau}_1^2[c_1\alpha\tau_1^2 + \tilde{\tau}_1^2(4c\kappa^2 + \lambda)]}{4c_2\alpha\tau_1^4}\right)t + \theta\right\}\right], \tag{134}$$

$$r(x,t) = \left\{\tilde{\tau}_0 + \tilde{\tau}_1\lambda_1 \pm \frac{\tilde{\tau}_1 W_4}{B(x + 2a\kappa t) - \xi_0}\right\}^2. \tag{135}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^3(\Psi - \lambda_2)$ and $\lambda_2 > \lambda_1$,

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_1 + \frac{4\tau_1 W_4^2(\lambda_2 - \lambda_1)}{4W_4^2 - [(\lambda_1 - \lambda_2)(B(x + 2a\kappa t) - \xi_0)]^2}\right\} \times \exp\left[i\left\{-\kappa x - \left(\frac{3b\tilde{\tau}_1^2[c_1\alpha\tau_1^2 + \tilde{\tau}_1^2(4c\kappa^2 + \lambda)]}{4c_2\alpha\tau_1^4}\right)t + \theta\right\}\right], \tag{136}$$

$$r(x,t) = \left\{\tilde{\tau}_0 + \tilde{\tau}_1\lambda_1 + \frac{4\tilde{\tau}_1 W_4^2(\lambda_2 - \lambda_1)}{4W_4^2 - [(\lambda_1 - \lambda_2)(B(x + 2a\kappa t) - \xi_0)]^2}\right\}^2. \tag{137}$$

When $\Lambda(\Psi) = (\Psi - \lambda_1)^2(\Psi - \lambda_2)^2$,

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_2 + \frac{\tau_1(\lambda_2 - \lambda_1)}{\exp\left[\frac{\lambda_1 - \lambda_2}{W_4}(B(x + 2a\kappa t) - \xi_0)\right] - 1}\right\} \times \exp\left[i\left\{-\kappa x - \left(\frac{3b\tilde{\tau}_1^2[c_1\alpha\tau_1^2 + \tilde{\tau}_1^2(4c\kappa^2 + \lambda)]}{4c_2\alpha\tau_1^4}\right)t + \theta\right\}\right], \tag{138}$$

$$r(x,t) = \left\{\tilde{\tau}_0 + \tilde{\tau}_1\lambda_2 + \frac{\tilde{\tau}_1(\lambda_2 - \lambda_1)}{\exp\left[\frac{\lambda_1 - \lambda_2}{W_4}(B(x + 2a\kappa t) - \xi_0)\right] - 1}\right\}^2, \tag{139}$$

and

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_1+\frac{\tau_1(\lambda_1-\lambda_2)}{\exp\left[\frac{\lambda_1-\lambda_2}{W_4}(B(x+2a\kappa t)-\xi_0)\right]-1}\right\}$$
$$\times\exp\left[i\left\{-\kappa x-\left(\frac{3b\tilde{\tau}_1^2[c_1\alpha\tau_1^2+\tilde{\tau}_1^2(4c\kappa^2+\lambda)]}{4c_2\alpha\tau_1^4}\right)t+\theta\right\}\right], \quad (140)$$

$$r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_1+\frac{\tilde{\tau}_1(\lambda_1-\lambda_2)}{\exp\left[\frac{\lambda_1-\lambda_2}{W_4}(B(x+2a\kappa t)-\xi_0)\right]-1}\right\}^2. \quad (141)$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)^2(\Psi-\lambda_2)(\Psi-\lambda_3)$ and $\lambda_1>\lambda_2>\lambda_3$,

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_1-\frac{2\tau_1(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{2\lambda_1-\lambda_2-\lambda_3+(\lambda_3-\lambda_2)\cosh\left[\frac{B\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{W_4}(x+2a\kappa t)\right]}\right\}$$
$$\times\exp\left[i\left\{-\kappa x-\left(\frac{3b\tilde{\tau}_1^2[c_1\alpha\tau_1^2+\tilde{\tau}_1^2(4c\kappa^2+\lambda)]}{4c_2\alpha\tau_1^4}\right)t+\theta\right\}\right], \quad (142)$$

$$r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_1-\frac{2\tilde{\tau}_1(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{2\lambda_1-\lambda_2-\lambda_3+(\lambda_3-\lambda_2)\cosh\left[\frac{B\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{W_4}(x+2a\kappa t)\right]}\right\}^2. \quad (143)$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)(\Psi-\lambda_2)(\Psi-\lambda_3)(\Psi-\lambda_4)$ and $\lambda_1>\lambda_2>\lambda_3>\lambda_4$,

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_2+\frac{\tau_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\mathrm{sn}^2\left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W_4}(B(x+2a\kappa t)-\xi_0),l\right]}\right\}$$
$$\times\exp\left[i\left\{-\kappa x-\left(\frac{3b\tilde{\tau}_1^2\left[c_1\alpha\tau_1^2+\tilde{\tau}_1^2(4c\kappa^2+\lambda)\right]}{4c_2\alpha\tau_1^4}\right)t+\theta\right\}\right], \quad (144)$$

$$r(x,t) = \left\{ \tilde{\tau}_0 + \tilde{\tau}_1\lambda_2 + \frac{\tilde{\tau}_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\mathrm{sn}^2 \times \left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W_4}(B(x+2a\kappa t)-\xi_0), l\right]} \right\}^2, \tag{145}$$

where

$$l^2 = \frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}. \tag{146}$$

Note that λ_i $(i = 1, \ldots, 4)$ are the roots of the polynomial equation

$$\Lambda(\Psi) = 0. \tag{147}$$

When $\tau_0 = -\tau_1\lambda_1$, $\tilde{\tau}_0 = -\tilde{\tau}_1\lambda_1$ and $\xi_0 = 0$, the solutions (134)–(143) can be reduced to plane wave solutions

$$q(x,t) = \left\{\pm\frac{\tau_1 W_4}{B(x+2a\kappa t)}\right\} \times \exp\left[i\left\{-\kappa x - \left(\frac{3b\tilde{\tau}_1^2[c_1\alpha\tau_1^2+\tilde{\tau}_1^2(4c\kappa^2+\lambda)]}{4c_2\alpha\tau_1^4}\right)t+\theta\right\}\right], \tag{148}$$

$$r(x,t) = \left\{\pm\frac{\tilde{\tau}_1 W_4}{B(x+2a\kappa t)}\right\}^2, \tag{149}$$

$$q(x,t) = \left\{\frac{4\tau_1 W_4^2(\lambda_2-\lambda_1)}{4W_4^2-[B(\lambda_1-\lambda_2)(x+2a\kappa t)]^2}\right\} \times \exp\left[i\left\{-\kappa x - \left(\frac{3b\tilde{\tau}_1^2[c_1\alpha\tau_1^2+\tilde{\tau}_1^2(4c\kappa^2+\lambda)]}{4c_2\alpha\tau_1^4}\right)t+\theta\right\}\right], \tag{150}$$

$$r(x,t) = \left\{\frac{4\tilde{\tau}_1 W_4^2(\lambda_2-\lambda_1)}{4W_4^2-[B(\lambda_1-\lambda_2)(x+2a\kappa t)]^2}\right\}^2, \tag{151}$$

singular soliton solutions

$$q(x,t) = \left\{\frac{\tau_1(\lambda_2-\lambda_1)}{2}\left(1\mp\coth\left[\frac{B(\lambda_1-\lambda_2)}{2W_4}(x+2a\kappa t)\right]\right)\right\} \times \exp\left[i\left\{-\kappa x - \left(\frac{3b\tilde{\tau}_1^2\left[c_1\alpha\tau_1^2+\tilde{\tau}_1^2(4c\kappa^2+\lambda)\right]}{4c_2\alpha\tau_1^4}\right)t+\theta\right\}\right], \tag{152}$$

$$r(x,t) = \left\{\frac{\tilde{\tau}_1(\lambda_2-\lambda_1)}{2}\left(1\mp\coth\left[\frac{B(\lambda_1-\lambda_2)}{2W_4}(x+2a\kappa t)\right]\right)\right\}^2, \tag{153}$$

and nematicon solutions

$$q(x,t) = \left\{\frac{A_7}{(C_3 + \cosh[B_7(x + 2a\kappa t)])}\right\} \times \exp\left[i\left\{-\kappa x - \left(\frac{3b\tilde{\tau}_1^2[c_1\alpha\tau_1^2 + \tilde{\tau}_1^2(4c\kappa^2 + \lambda)]}{4c_2\alpha\tau_1^4}\right)t + \theta\right\}\right], \quad (154)$$

$$r(x,t) = \left\{\frac{\widetilde{A}_7}{(C_3 + \cosh[B_7(x + 2a\kappa t)])^2}\right\}, \quad (155)$$

where

$$A_7 = \frac{2\tau_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}{\lambda_3 - \lambda_2}, \quad \widetilde{A}_7 = \left(\frac{2\tilde{\tau}_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}{\lambda_3 - \lambda_2}\right)^2,$$
$$B_7 = \frac{B\sqrt{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}}{W_4}, \quad C_3 = \frac{2\lambda_1 - \lambda_2 - \lambda_3}{\lambda_3 - \lambda_2}. \quad (156)$$

Here, A_7 and $\widetilde{A}_7$ are the amplitudes of the solitons, while B_7 is the inverse width of the solitons. These solitons are valid for $\tau_1 < 0$ and $\tilde{\tau}_1 < 0$. Moreover, when $\tau_0 = -\tau_1\lambda_2$ and $\xi_0 = 0$, Jacobi elliptic function solutions (144) and (145) can be reduced as follows:

$$q(x,t) = \left\{\frac{A_8}{C_4 + \mathrm{sn}^2\left[B_j(x + 2a\kappa t), \frac{(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_4)}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}\right]}\right\} \times \exp\left[i\left\{-\kappa x - \left(\frac{3b\tilde{\tau}_1^2[c_1\alpha\tau_1^2 + \tilde{\tau}_1^2(4c\kappa^2 + \lambda)]}{4c_2\alpha\tau_1^4}\right)t + \theta\right\}\right], \quad (157)$$

$$r(x,t) = \left\{\frac{\widetilde{A}_8}{\left(C_4 + \mathrm{sn}^2\left[B_j(x + 2a\kappa t), \frac{(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_4)}{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}\right]\right)^2}\right\}, \quad (158)$$

where

$$A_8 = \frac{\tau_1(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_1 - \lambda_4}, \quad \widetilde{A}_8 = \left(\frac{\tilde{\tau}_1(\lambda_1 - \lambda_2)(\lambda_4 - \lambda_2)}{\lambda_1 - \lambda_4}\right)^2,$$
$$C_4 = \frac{\lambda_4 - \lambda_2}{\lambda_1 - \lambda_4}, \quad B_j = \frac{(-1)^j B\sqrt{(\lambda_1 - \lambda_3)(\lambda_2 - \lambda_4)}}{2W_4}, \quad (j = 8, 9). \quad (159)$$

Remark 6. When the modulus $l \to 1$, the following singular soliton solutions evolve:

$$q(x,t) = \left\{\frac{A_8}{C_4 + \tanh^2[B_j(x + 2a\kappa t)]}\right\} \times \exp\left[i\left\{-\kappa x - \left(\frac{3b\tilde{\tau}_1^2[c_1\alpha\tau_1^2 + \tilde{\tau}_1^2(4c\kappa^2 + \lambda)]}{4c_2\alpha\tau_1^4}\right)t + \theta\right\}\right], \quad (160)$$

$$r(x,t) = \left\{ \frac{\widetilde{A}_8}{(C_4 + \tanh^2[B_j(x + 2a\kappa t)])^2} \right\}, \tag{161}$$

where $\lambda_3 = \lambda_4$.

Remark 7. However, if $l \to 0$, periodic singular wave solutions are as follows:

$$q(x,t) = \left\{ \frac{A_8}{C_4 + \sin^2[B_j(x + 2a\kappa t)]} \right\} \times \exp\left[i \left\{ -\kappa x - \left(\frac{3b\tilde{\tau}_1^2[c_1\alpha\tau_1^2 + \tilde{\tau}_1^2(4c\kappa^2 + \lambda)]}{4c_2\alpha\tau_1^4} \right) t + \theta \right\} \right], \tag{162}$$

$$r(x,t) = \left\{ \frac{\widetilde{A}_8}{(C_4 + \sin^2[B_j(x + 2a\kappa t)])^2} \right\}, \tag{163}$$

where $\lambda_2 = \lambda_3$.

5. Dual-Power Law

For dual-power law, the nonlinearity is structured as

$$F(s) = c_1 s^n + c_2 s^{2n}, \tag{164}$$

so that Eq. (2) modifies to

$$cr_{xx} + \lambda r + \alpha(c_1|q|^{2n} + c_2|q|^{4n}) = 0. \tag{165}$$

In this case, Eq. (9) simplifies to

$$cB^2 Q'' + \lambda Q + \alpha(c_1 P^{2n} + c_2 P^{4n}) = 0. \tag{166}$$

In order to obtain closed form solutions, we use the transformation

$$P = U_1^{\frac{1}{n}} \quad \text{and} \quad Q = U_2^2, \tag{167}$$

that will reduce Eqs. (8) and (166) into the ODEs

$$-(\omega + a\kappa^2)U_1^2 + aB^2 \left\{ \frac{1-n}{n^2}(U_1')^2 + \frac{1}{n}U_1 U_1'' \right\} + bU_1^2 U_2^2 = 0, \tag{168}$$

and

$$\alpha(c_1 U_1^2 + c_2 U_1^4) + \lambda U_2^2 + 2cB^2\{(U_2')^2 + U_2 U_2''\} = 0. \tag{169}$$

By extended trial equation method, Eqs. (168) and (169) will now be analyzed in the remaining section for soliton and other solutions. Using the balance principle, one finds that

$$\varsigma = \tilde{\varsigma} = \frac{\sigma - \rho - 2}{2}. \tag{170}$$

Let us choose $\sigma = 4$, $\rho = 0$ and $\varsigma = \tilde{\varsigma} = 1$ in Eq. (170). Then, substituting the terms in Eqs. 13)–(19) into Eqs. (168) and (169), and solving the resulting system of algebraic equations, one recovers the following results:

$$\mu_0 = -\frac{\chi_0\tilde{\tau}_0^2\left[c_2\tau_1^2(a\lambda(1+n)+4bcn^2\tilde{\tau}_0^2)+6bcc_1n^2\tilde{\tau}_1^2\right]}{4acc_2\tau_1^2\tilde{\tau}_1^2B^2(1+n)},$$
$$\mu_1 = -\frac{\chi_0\tilde{\tau}_0\left[c_2\tau_1^2(a\lambda(1+n)+8bcn^2\tilde{\tau}_0^2)+6bcc_1n^2\tilde{\tau}_1^2\right]}{2acc_2\tau_1^2\tilde{\tau}_1B^2(1+n)},$$
$$\mu_2 = -\frac{\chi_0\left[c_2\tau_1^2(a\lambda(1+n)+24bcn^2\tilde{\tau}_0^2)+6bcc_1n^2\tilde{\tau}_1^2\right]}{4acc_2\tau_1^2B^2(1+n)}, \tag{171}$$
$$\mu_3 = -\frac{4bn^2\chi_0\tilde{\tau}_0\tilde{\tau}_1}{aB^2(1+n)},\quad \mu_4 = -\frac{bn^2\chi_0\tilde{\tau}_1^2}{aB^2(1+n)},\quad \tau_0 = \frac{\tau_1\tilde{\tau}_0}{\tilde{\tau}_1},$$
$$\alpha = \frac{6bcn^2\tilde{\tau}_1^4}{ac_2\tau_1^4(1+n)},\quad \omega = -\frac{6bcc_1n^2\tilde{\tau}_1^2+ac_2\tau_1^2(1+n)(4cn^2\kappa^2+\lambda)}{4cc_2n^2\tau_1^2(1+n)},$$
$$\chi_0=\chi_0,\quad \tau_1=\tau_1,\quad \tilde{\tau}_0=\tilde{\tau}_0,\quad \tilde{\tau}_1=\tilde{\tau}_1.$$

Substituting these results into Eqs. (15) and (20), we find that

$$\pm(\xi-\xi_0) = W_5\int\frac{d\Psi}{\sqrt{\Lambda(\Psi)}}, \tag{172}$$

where

$$\Lambda(\Psi) = \Psi^4+\frac{\mu_3}{\mu_4}\Psi^3+\frac{\mu_2}{\mu_4}\Psi^2+\frac{\mu_1}{\mu_4}\Psi+\frac{\mu_0}{\mu_4},\quad W_5=\sqrt{\frac{\chi_0}{\mu_4}}. \tag{173}$$

Consequently, one recovers the traveling wave solutions to (1) and (165) as follows:
When $\Lambda(\Psi) = (\Psi-\lambda_1)^4$,

$$q(x,t) = \left\{\tau_0+\tau_1\lambda_1\pm\frac{\tau_1W_5}{B(x+2a\kappa t)-\xi_0}\right\}^{\frac{1}{n}} \times\exp\left[i\left\{-\kappa x-\left(\frac{6bcc_1n^2\tilde{\tau}_1^2+ac_2\tau_1^2(1+n)(4cn^2\kappa^2+\lambda)}{4cc_2n^2\tau_1^2(1+n)}\right)t+\theta\right\}\right], \tag{174}$$

$$r(x,t) = \left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_1\pm\frac{\tilde{\tau}_1W_5}{B(x+2a\kappa t)-\xi_0}\right\}^2. \tag{175}$$

When $\Lambda(\Psi) = (\Psi-\lambda_1)^3(\Psi-\lambda_2)$ and $\lambda_2>\lambda_1$,

$$q(x,t) = \left\{\tau_0+\tau_1\lambda_1+\frac{4\tau_1W_5^2(\lambda_2-\lambda_1)}{4W_5^2-[(\lambda_1-\lambda_2)(B(x+2a\kappa t)-\xi_0)]^2}\right\}^{\frac{1}{n}} \times\exp\left[i\left\{-\kappa x-\left(\frac{6bcc_1n^2\tilde{\tau}_1^2+ac_2\tau_1^2(1+n)(4cn^2\kappa^2+\lambda)}{4cc_2n^2\tau_1^2(1+n)}\right)t+\theta\right\}\right], \tag{176}$$

$$r(x,t) = \left\{\tilde{\tau}_0 + \tilde{\tau}_1\lambda_1 + \frac{4\tilde{\tau}_1 W_5^2(\lambda_2-\lambda_1)}{4W_5^2 - [(\lambda_1-\lambda_2)(B(x+2a\kappa t)-\xi_0)]^2}\right\}^2. \tag{177}$$

When $\Lambda(\Psi) = (\Psi-\lambda_1)^2(\Psi-\lambda_2)^2$,

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_2 + \frac{\tau_1(\lambda_2-\lambda_1)}{\exp\left[\frac{\lambda_1-\lambda_2}{W_5}(B(x+2a\kappa t)-\xi_0)\right]-1}\right\}^{\frac{1}{n}} \times \exp\left[i\left\{-\kappa x - \left(\frac{6bcc_1n^2\tilde{\tau}_1^2 + ac_2\tau_1^2(1+n)(4cn^2\kappa^2+\lambda)}{4cc_2n^2\tau_1^2(1+n)}\right)t+\theta\right\}\right], \tag{178}$$

$$r(x,t) = \left\{\tilde{\tau}_0 + \tilde{\tau}_1\lambda_2 + \frac{\tilde{\tau}_1(\lambda_2-\lambda_1)}{\exp\left[\frac{\lambda_1-\lambda_2}{W_5}(B(x+2a\kappa t)-\xi_0)\right]-1}\right\}^2, \tag{179}$$

and

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_1 + \frac{\tau_1(\lambda_1-\lambda_2)}{\exp\left[\frac{\lambda_1-\lambda_2}{W_5}(B(x+2a\kappa t)-\xi_0)\right]-1}\right\}^{\frac{1}{n}} \times \exp\left[i\left\{-\kappa x - \left(\frac{6bcc_1n^2\tilde{\tau}_1^2 + ac_2\tau_1^2(1+n)(4cn^2\kappa^2+\lambda)}{4cc_2n^2\tau_1^2(1+n)}\right)t+\theta\right\}\right], \tag{180}$$

$$r(x,t) = \left\{\tilde{\tau}_0 + \tilde{\tau}_1\lambda_1 + \frac{\tilde{\tau}_1(\lambda_1-\lambda_2)}{\exp\left[\frac{\lambda_1-\lambda_2}{W_5}(B(x+2a\kappa t)-\xi_0)\right]-1}\right\}^2. \tag{181}$$

When $\Lambda(\Psi) = (\Psi-\lambda_1)^2(\Psi-\lambda_2)(\Psi-\lambda_3)$ and $\lambda_1 > \lambda_2 > \lambda_3$,

$$q(x,t) = \left\{\tau_0 + \tau_1\lambda_1 - \frac{2\tau_1(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{2\lambda_1-\lambda_2-\lambda_3+(\lambda_3-\lambda_2)\cosh \times \left[\frac{B\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{W_5}(x+2a\kappa t)\right]}\right\}^{\frac{1}{n}} \times \exp\left[i\left\{-\kappa x - \left(\frac{6bcc_1n^2\tilde{\tau}_1^2 + ac_2\tau_1^2(1+n)(4cn^2\kappa^2+\lambda)}{4cc_2n^2\tau_1^2(1+n)}\right)t+\theta\right\}\right], \tag{182}$$

$$r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_1-\frac{2\tilde{\tau}_1(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{2\lambda_1-\lambda_2-\lambda_3+(\lambda_3-\lambda_2)\cosh\left[\frac{B\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}}{W_5}(x+2a\kappa t)\right]}\right\}^2. \tag{183}$$

When $\Lambda(\Psi)=(\Psi-\lambda_1)(\Psi-\lambda_2)(\Psi-\lambda_3)(\Psi-\lambda_4)$ and $\lambda_1>\lambda_2>\lambda_3>\lambda_4$,

$$q(x,t)=\left\{\tau_0+\tau_1\lambda_2+\frac{\tau_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\mathrm{sn}^2\left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W_5}(B(x+2a\kappa t)-\xi_0),l\right]}\right\}^{\frac{1}{n}}$$
$$\times\exp\left[i\left\{-\kappa x-\left(\frac{6bcc_1n^2\tilde{\tau}_1^2+ac_2\tau_1^2(1+n)(4cn^2\kappa^2+\lambda)}{4cc_2n^2\tau_1^2(1+n)}\right)t+\theta\right\}\right], \tag{184}$$

$$r(x,t)=\left\{\tilde{\tau}_0+\tilde{\tau}_1\lambda_2+\frac{\tilde{\tau}_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_4-\lambda_2+(\lambda_1-\lambda_4)\mathrm{sn}^2\left[\pm\frac{\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W_5}(B(x+2a\kappa t)-\xi_0),l\right]}\right\}^2, \tag{185}$$

where

$$l^2=\frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}. \tag{186}$$

Note that λ_i $(i=1,\ldots,4)$ are the roots of the polynomial equation

$$\Lambda(\Psi)=0. \tag{187}$$

When $\tau_0=-\tau_1\lambda_1$, $\tilde{\tau}_0=-\tilde{\tau}_1\lambda_1$ and $\xi_0=0$, the solutions (174)–(183) can be reduced to plane wave solutions

$$q(x,t)=\left\{\pm\frac{\tau_1W_5}{B(x+2a\kappa t)}\right\}^{\frac{1}{n}}$$
$$\times\exp\left[i\left\{-\kappa x-\left(\frac{6bcc_1n^2\tilde{\tau}_1^2+ac_2\tau_1^2(1+n)(4cn^2\kappa^2+\lambda)}{4cc_2n^2\tau_1^2(1+n)}\right)t+\theta\right\}\right], \tag{188}$$

$$r(x,t) = \left\{ \pm \frac{\tilde{\tau}_1 W_5}{B(x+2a\kappa t)} \right\}^2, \tag{189}$$

$$q(x,t) = \left\{ \frac{4\tau_1 W_5^2(\lambda_2 - \lambda_1)}{4W_5^2 - [B(\lambda_1 - \lambda_2)(x+2a\kappa t)]^2} \right\}^{\frac{1}{n}} \times \exp\left[i \left\{ -\kappa x - \left(\frac{6bcc_1 n^2 \tilde{\tau}_1^2 + ac_2\tau_1^2(1+n)(4cn^2\kappa^2 + \lambda)}{4cc_2 n^2 \tau_1^2(1+n)} \right) t + \theta \right\} \right], \tag{190}$$

$$r(x,t) = \left\{ \frac{4\tilde{\tau}_1 W_5^2(\lambda_2 - \lambda_1)}{4W_5^2 - [B(\lambda_1 - \lambda_2)(x+2a\kappa t)]^2} \right\}^2, \tag{191}$$

singular soliton solutions

$$q(x,t) = \left\{ \frac{\tau_1(\lambda_2 - \lambda_1)}{2} \left(1 \mp \coth\left[\frac{B(\lambda_1 - \lambda_2)}{2W_5}(x+2a\kappa t) \right] \right) \right\}^{\frac{1}{n}} \times \exp\left[i \left\{ -\kappa x - \left(\frac{6bcc_1 n^2 \tilde{\tau}_1^2 + ac_2\tau_1^2(1+n)(4cn^2\kappa^2 + \lambda)}{4cc_2 n^2 \tau_1^2(1+n)} \right) t + \theta \right\} \right], \tag{192}$$

$$r(x,t) = \left\{ \frac{\tilde{\tau}_1(\lambda_2 - \lambda_1)}{2} \left(1 \mp \coth\left[\frac{B(\lambda_1 - \lambda_2)}{2W_5}(x+2a\kappa t) \right] \right) \right\}^2, \tag{193}$$

and nematicon solutions

$$q(x,t) = \left\{ \frac{A_9}{(C_5 + \cosh[B_{10}(x+2a\kappa t)])^{\frac{1}{n}}} \right\} \times \exp\left[i \left\{ -\kappa x - \left(\frac{6bcc_1 n^2 \tilde{\tau}_1^2 + ac_2\tau_1^2(1+n)(4cn^2\kappa^2 + \lambda)}{4cc_2 n^2 \tau_1^2(1+n)} \right) t + \theta \right\} \right], \tag{194}$$

$$r(x,t) = \left\{ \frac{\widetilde{A}_9}{(C_5 + \cosh[B_{10}(x+2a\kappa t)])^2} \right\}, \tag{195}$$

where

$$A_9 = \left(\frac{2\tau_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}{\lambda_3 - \lambda_2} \right)^{\frac{1}{n}}, \quad \widetilde{A}_9 = \left(\frac{2\tilde{\tau}_1(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}{\lambda_3 - \lambda_2} \right)^2, \quad B_{10} = \frac{B\sqrt{(\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)}}{W_5}, \quad C_5 = \frac{2\lambda_1 - \lambda_2 - \lambda_3}{\lambda_3 - \lambda_2}. \tag{196}$$

Here, A_9 and $\widetilde{A}_9$ are the amplitudes of the solitons, while B_{10} is the inverse width of the solitons. These solitons are valid for $\tau_1 < 0$ and $\tilde{\tau}_1 < 0$. Moreover, when $\tau_0 = -\tau_1\lambda_2$ and $\xi_0 = 0$, Jacobi elliptic function solutions (184) and (185) can be written as

$$q(x,t)=\left\{\frac{A_{10}}{\left(C_6+\mathrm{sn}^2\left[B_j(x+2a\kappa t),\frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}\right]\right)^{\frac{1}{n}}}\right\}$$
$$\times\exp\left[i\left\{-\kappa x-\left(\frac{6bcc_1n^2\tilde{\tau}_1^2+ac_2\tau_1^2(1+n)(4cn^2\kappa^2+\lambda)}{4cc_2n^2\tau_1^2(1+n)}\right)t+\theta\right\}\right], \tag{197}$$

$$r(x,t)=\left\{\frac{\widetilde{A}_{10}}{\left(C_6+\mathrm{sn}^2\left[B_j(x+2a\kappa t),\frac{(\lambda_2-\lambda_3)(\lambda_1-\lambda_4)}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}\right]\right)^{2}}\right\}, \tag{198}$$

where

$$A_{10}=\left(\frac{\tau_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_1-\lambda_4}\right)^{\frac{1}{n}},\quad \widetilde{A}_{10}=\left(\frac{\tilde{\tau}_1(\lambda_1-\lambda_2)(\lambda_4-\lambda_2)}{\lambda_1-\lambda_4}\right)^{2},$$
$$C_6=\frac{\lambda_4-\lambda_2}{\lambda_1-\lambda_4},\quad B_j=\frac{(-1)^jB\sqrt{(\lambda_1-\lambda_3)(\lambda_2-\lambda_4)}}{2W_5},\ (j=11,12). \tag{199}$$

Remark 8. When the modulus $l\to 1$, the following singular solitons are listed:

$$q(x,t)=\left\{\frac{A_{10}}{(C_6+\tanh^2[B_j(x+2a\kappa t)])^{\frac{1}{n}}}\right\}$$
$$\times\exp\left[i\left\{-\kappa x-\left(\frac{6bcc_1n^2\tilde{\tau}_1^2+ac_2\tau_1^2(1+n)(4cn^2\kappa^2+\lambda)}{4cc_2n^2\tau_1^2(1+n)}\right)t+\theta\right\}\right], \tag{200}$$

$$r(x,t)=\left\{\frac{\widetilde{A}_{10}}{(C_6+\tanh^2[B_j(x+2a\kappa t)])^2}\right\}, \tag{201}$$

where $\lambda_3=\lambda_4$.

Remark 9. However, if $l\to 0$, periodic singular wave solutions are listed as

$$q(x,t)=\left\{\frac{A_{10}}{(C_6+\sin^2[B_j(x+2a\kappa t)])^{\frac{1}{n}}}\right\}$$
$$\times\exp\left[i\left\{-\kappa x-\left(\frac{6bcc_1n^2\tilde{\tau}_1^2+ac_2\tau_1^2(1+n)(4cn^2\kappa^2+\lambda)}{4cc_2n^2\tau_1^2(1+n)}\right)t+\theta\right\}\right], \tag{202}$$

$$r(x,t)=\left\{\frac{\widetilde{A}_{10}}{(C_6+\sin^2[B_j(x+2a\kappa t)])^2}\right\}, \tag{203}$$

where $\lambda_2=\lambda_3$.

6. Conclusions

This paper carried out a comprehensive study of nematicons in liquid crystals that is considered with four forms of nonlinearity. The applied algorithm, namely the extended trial equation method, gave way to several other forms of waves in addition to nematicons. These are plane waves, shock waves, periodic singular waves, singular solitons and snoidal waves. The special cases of these doubly periodic functions are also listed based on the limiting value of the modulus of ellipticity. These plethora of wave solutions in liquid crystals are being reported for the first time in the context of liquid crystals. The results of this paper thus carry a lot of promise. The model needs to be studied, amongst other forms, with time-dependent coeficients as well as random coefficients. Such results are all awaited at this time.

Acknowledgments

The fifth author (QZ) was funded by the National Science Foundation of Hubei Province in China under the grant number 2015CFC891. The sixth author (SPM) would like to thank the research support provided by the Department of Mathematics and Statistics at Tshwane University of Technology and the support from the South African National Foundation under Grant Number 92052 IRF1202210126. The seventh author (AB) would like to thank Tshwane University of Technology during his academic visit during 2016. The research work of eighth author (MB) was supported by Qatar National Research Fund (QNRF) under the grant number NPRP 6-021-1-005. The authors also declare that there is no conflict of interest.

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