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Optical solitons in fiber Bragg gratings with generalized anti-cubic nonlinearity by extended auxiliary equation

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ABSTRACT

This work retrieves optical solitons having generalized anti-cubic nonlinearity in fiber Bragg gratings by implementing extended auxiliary equation method. The spectrum of solitons are enumerated along with their existence criteria.

1. Introduction

Optical solitons in fiber Bragg gratings (FBs) have attracted considerable attention since its first appearance about a couple of decades ago [1–20] The majority of the reported results in this context are from numerical perspective. Very recently, analytical results from this avenue have started pouring in [17–19] These results are with very limited forms of nonlinear refractive index. To enumerate, it is Kerr law, parabolic law, quadratic–cubic law, polynomial law and parabolic–nonlocal combo nonlinearity where substantial results with BGs have been reported. The current paper exhibits, for the first time, exact soliton solutions in fibers BGs that comes with generalized anti-cubic (AC) form of nonlinear refractive index. There are three values of the nonlinearity parameter as deemed permissible according to its domain of validity. The extended auxiliary equation approach first retrieves the solutions for the three models in terms of Jacobi's elliptic functions. Subsequently, in the limiting situation, when the modulus of ellipticity approaches zero or unity, it is the soliton solutions that emerge from the scheme. The details are outlined in subsequent sections after

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the three models are pen-pictured in the following subsection.

1.1. Governing model

The governing generalized AC nonlinearity is structured as [8,9,20,31–34]

$$iq_t + aq_{xx} + \left(\frac{b_1}{|q|^{2n+2}} + b_2|q|^{2n} + b_3|q|^{2n+2}\right)q = 0, \tag{1}$$

with $i = \sqrt{-1}$. Here, $q(x, t)$ is the complex-valued function that represent pulses transmitting across the fibers. In Eq. (1), the first term is the linear temporal evolution, while the coefficient a represents chromatic dispersion (CD) and the constant coefficients b_j for $j = 1, 2, 3$ give self-phase modulation (SPM). Also, n is the power-law nonlinearity parameter such that $-1 < n < 3$ [8]. The following subsections will introduce generalized AC nonlinearity in fiber BGs with three cases at $n = 0$, $n = 1$ and $n = 2$ where details are displayed.

1.1.1. Case-I ($n = 0$)

For optical fibers with differential group delay, generalized AC splits into two components with BGs at $n = 0$. Thus, the vector-coupled model reads

$$iu_t + a_1v_{xx} + \frac{f_1u}{b_1|u|^2 + c_1|v|^2} + d_1u + (\xi_1|u|^2 + \eta_1|v|^2)u + i\alpha_1u_x + \beta_1v = 0, \tag{2}$$

and

$$iv_t + a_2u_{xx} + \frac{f_2v}{b_2|v|^2 + c_2|u|^2} + d_2v + (\xi_2|v|^2 + \eta_2|u|^2)v + i\alpha_2v_x + \beta_2u = 0, \tag{3}$$

where a_l, f_l, b_l, c_l, d_l, ξ_l, η_l, α_l and β_l for $l = 1, 2$ are constants. The independent variables x and t represent spatial and temporal variables, respectively and the dependent variables $u(x, t)$ and $v(x, t)$ are wave profiles along the two components. The coefficients a_l represent dispersive reflectivity, while the coefficient f_l represents the combination of SPM and cross phase modulation effects (XPM). The coefficients b_l, d_l and ξ_l give SPM and the coefficients c_l and η_l are XPM. Next, the coefficient α_l are from inter-modal dispersion and finally, β_l accounts for detuning.

1.1.2. Case-II ($n = 1$)

For $n = 1$, generalized AC splits into two components so that the vector-coupled model reads:

$$iu_t + a_1v_{xx} + \frac{f_1u}{b_1|u|^4 + c_1|u|^2|v|^2 + d_1|v|^4} + (\xi_1|u|^2 + \eta_1|v|^2)u + (\theta_1|u|^4 + \gamma_1|u|^2|v|^2 + \delta_1|v|^4)u + i\alpha_1u_x + \beta_1v = 0, \tag{4}$$

and

$$iv_t + a_2u_{xx} + \frac{f_2v}{b_2|v|^4 + c_2|v|^2|u|^2 + d_2|u|^4} + (\xi_2|v|^2 + \eta_2|u|^2)v + (\theta_2|v|^4 + \gamma_2|v|^2|u|^2 + \delta_2|u|^4)v + i\alpha_2v_x + \beta_2u = 0, \tag{5}$$

where a_l, f_l, b_l, c_l, d_l, ξ_l, η_l, θ_l, γ_l, δ_l, α_l and β_l for $l = 1, 2$ are constants. The coefficients b_l, ξ_l and θ_l are from SPM and the coefficients c_l, d_l, η_l and δ_l are associated with XPM.

1.1.3. Case-III ($n = 2$)

For BGs with $n = 2$, the vector-coupled NLDSE with generalized AC nonlinearity is casted as:

$$\begin{aligned} iu_t + a_1v_{xx} &+ \frac{f_1u}{b_1|u|^6 + c_1|u|^4|v|^2 + d_1|u|^2|v|^4 + e_1|v|^6} + (\theta_1|u|^4 + \gamma_1|u|^2|v|^2 + \delta_1|v|^4)u \\ &+ (\xi_1|u|^6 + \sigma_1|u|^4|v|^2 + r_1|u|^2|v|^4 + \eta_1|v|^6)u + i\alpha_1u_x + \beta_1v = 0, \end{aligned} \tag{6}$$

and

$$\begin{aligned} iv_t + a_2u_{xx} &+ \frac{f_2v}{b_2|v|^6 + c_2|v|^4|u|^2 + d_2|v|^2|u|^4 + e_2|u|^6} + (\theta_2|v|^4 + \gamma_2|v|^2|u|^2 + \delta_2|u|^4)v \\ &+ (\xi_2|v|^6 + \sigma_2|v|^4|u|^2 + r_2|v|^2|u|^4 + \eta_2|u|^6)v + i\alpha_2v_x + \beta_2u = 0, \end{aligned} \tag{7}$$

where a_l, f_l, b_l, c_l, d_l, e_l, θ_l, γ_l, δ_l, ξ_l, σ_l, r_l, η_l, α_l and β_l for $l = 1, 2$ are constants. Here, the coefficients f_l, c_l, γ_l, σ_l and r_l collectively represent SPM and XPM. The constants b_l, θ_l and ξ_l come from SPM and d_l, e_l, δ_l and η_l emanate from XPM.

The objective of this paper is to construct solutions to the above coupled systems in terms of Jacobi's elliptic functions. In the limiting case of the modulus of ellipticity, optical solitons and other solutions will emerge. To our best knowledge, the above coupled systems were not reported elsewhere, thus far.

This paper is structured as follows: In Sections 2, 3 and 4, the extended auxiliary equation method is implemented for the three cases of generalized AC nonlinearity applicable to fiber BGs. A wide range of solutions in terms of Jacobi's elliptic functions are enumerated. Subsequently, solitons and other solutions, that emerge from the limiting process are presented.

2. Mathematical analysis: Case-I ($n = 0$)

For the exact soliton solutions of the coupled system Eqs. (2) and (3), we introduce the transformation:

$$u(x, t) = \varphi_1(\xi)\exp[i\eta(x, t)],\\ v(x, t) = \varphi_2(\xi)\exp[i\eta(x, t)], \tag{8}$$

and

$$\xi = x - \upsilon t, \qquad \eta(x, t) = -kx + \omega t + \theta_0, \tag{9}$$

where υ, k, ω and θ_0 are all non zero constants to be determined which represent velocity of soliton, frequency of soliton, wave number and phase constant, respectively, while $\varphi_1(\xi)$, $\varphi_2(\xi)$ and $\eta(x, t)$ are real functions. Substituting Eq. (8) and (9) into the system Eq. (2) and (3), separating the real and the imaginary parts of the system Eq. (2) and (3), we have:

$$a_1\varphi_2'' - (\omega - d_1 - k\alpha_1)\varphi_1 - (a_1k^2 - \beta_1)\varphi_2 + \frac{f_1\varphi_1}{b_1\varphi_1^2 + c_1\varphi_2^2} + (\xi_1\varphi_1^2 + \eta_1\varphi_2^2)\varphi_1 = 0, \tag{10}$$

$$a_2\varphi_1'' - (\omega - d_2 - k\alpha_2)\varphi_2 - (a_2k^2 - \beta_2)\varphi_1 + \frac{f_2\varphi_2}{b_2\varphi_2^2 + c_2\varphi_1^2} + (\xi_2\varphi_2^2 + \eta_2\varphi_1^2)\varphi_2 = 0, \tag{11}$$

and

$$(\upsilon - \alpha_1)\varphi_1' + 2a_1k\varphi_2' = 0, \tag{12}$$

$$2a_2k\varphi_1' + (\upsilon - \alpha_2)\varphi_2' = 0. \tag{13}$$

Setting

$$\varphi_2(\xi) = \lambda_1\varphi_1(\xi), \tag{14}$$

where λ_1 is a non zero constant, such that $\lambda_1 \neq 1$. Consequently, Eqs. (10)–(13) reduce to

$$a_1\lambda_1\varphi_1'' - [\omega - d_1 - k\alpha_1 + \lambda_1(a_1k^2 - \beta_1)]\varphi_1 + \frac{f_1}{(b_1 + \lambda_1^2c_1)\varphi_1} + (\xi_1 + \lambda_1^2\eta_1)\varphi_1^3 = 0, \tag{15}$$

$$a_2\varphi_1'' - [\lambda_1(\omega - d_2 - k\alpha_2) + a_2k^2 - \beta_2]\varphi_1 + \frac{\lambda_1f_2}{(\lambda_1^2b_2 + c_2)\varphi_1} + \lambda_1(\lambda_1^2\xi_2 + \eta_2)\varphi_1^3 = 0, \tag{16}$$

and

$$(\upsilon - \alpha_1 + 2a_1k\lambda_1)\varphi_1' = 0, \tag{17}$$

$$[2a_2k + \lambda_1(\upsilon - \alpha_2)]\varphi_1' = 0. \tag{18}$$

From Eqs. (17) and (18), one can obtain the velocity of the soliton as

$$\upsilon = \alpha_1 - 2a_1k\lambda_1, \tag{19}$$

and

$$\upsilon = \frac{\alpha_2\lambda_1 - 2a_2k}{\lambda_1}. \tag{20}$$

From Eqs. (19) and (20), we have the constraint condition:

$$2a_1k\lambda_1^2 + (\alpha_2 - \alpha_1)\lambda_1 - 2a_2k = 0. \tag{21}$$

From (21), we have

$$\lambda_1 = \frac{-(\alpha_2 - \alpha_1) \pm \sqrt{(\alpha_2 - \alpha_1)^2 + 16a_1a_2k^2}}{4a_1k}, \tag{22}$$

provided $a_1a_2 > 0$. Eqs. (15) and (16) have the same form under the constraint conditions:

$$a_1\lambda_1 = a_2, \tag{23}$$

$$(\xi_1 + \lambda_1^2\eta_1) = \lambda_1(\lambda_1^2\xi_2 + \eta_2), \tag{24}$$

$$f_1(\lambda_1^2b_2 + c_2) = \lambda_1f_2(b_1 + \lambda_1^2c_1), \tag{25}$$

$$[\omega - d_1 - k\alpha_1 + \lambda_1(a_1k^2 - \beta_1)] = [\lambda_1(\omega - d_2 - k\alpha_2) + a_2k^2 - \beta_2]. \tag{26}$$

Eq. (15) can be rewritten in the form:

$$\varphi_1\varphi_1'' + \rho_1 - \rho_2\varphi_1^2 + \rho_3\varphi_1^4 = 0, \tag{27}$$

where

$$\rho_1 = \frac{f_1}{a_1\lambda_1(b_1 + \lambda_1^2c_1)},\ \rho_2 = \frac{[\omega - d_1 - k\alpha_1 + \lambda_1(a_1k^2 - \beta_1)]}{a_1\lambda_1},\ \rho_3 = \frac{(\xi_1 + \lambda_1^2\eta_1)}{a_1\lambda_1}, \tag{28}$$

provided $a_1\lambda_1(b_1 + \lambda_1^2c_1) \neq 0$. Balancing [44–58] $\varphi_1(\xi)\varphi_1''(\xi)$ with $\varphi_1^4(\xi)$ in Eq. (27)yields $N = 1$. In the next subsection, we solve Eq. (27) using the proposed method.

2.1. Extended auxiliary equation

According to this method [19,20], we assume that Eq. (27) has the formal solution:

$$\varphi_1(\xi) = A_0 + A_1F(\xi) + A_2F^2(\xi), \tag{29}$$

where A_0, A_1 and A_2 are constants to be determined, such that $A_2 \neq 0$, while the function $F(\xi)$ satisfies the following first order equation:

$$F'^2(\xi) = c_0 + c_2F^2(\xi) + c_4F^4(\xi) + c_6F^6(\xi), \tag{30}$$

where $c_j (j = 0, 2, 4, 6)$ are constants to be determined. It is well known that Eq. (30) has the following solution:

$$F(\xi) = \frac{1}{2}\left[-\frac{c_4}{c_6}(1 \pm f(\xi))\right]^{\frac{1}{2}}, \tag{31}$$

where $f(\xi)$ could be expressed through the Jacobi elliptic functions sn(ξ, m), cn(ξ, m), dn(ξ, m) and so on. Here $0 < m < 1$ is the modulus of the Jacobi elliptic functions. Substituting Eq. (29) along with Eq. (30) into Eq. (27), collecting the coefficients of each power $F^i(\xi)(F'(\xi))^j$, $(i = 0, 1, 2, ..,8,\ \ j = 0, 1)$, and setting these coefficients to zero, we have a set of algebraic equations which can be solved by the aid of Maple to obtain the following results:

$$c_0 = -\frac{A_0(A_0^2\rho_3 - \rho_2)}{2A_2},\ c_2 = -\frac{3}{4}A_0^2\rho_3 + \frac{1}{4}\rho_2,\ c_4 = -\frac{1}{2}A_0A_2\rho_3,\ c_6 = -\frac{1}{8}A_2^2\rho_3,$$
$$A_0 = A_0,\ A_1 = 0,\ A_2 = A_2,\ \rho_1 = 0,\ \rho_2 = \rho_2,\ \rho_3 = \rho_3. \tag{32}$$

From Eqs. (29), (31) and (32), then we have the solutions:

$$\varphi_1(\xi) = \pm A_0\ f(\xi). \tag{33}$$

The coupled system (2) and ((3) has the following families of solutions:

Family-1: If $c_0 = \frac{c_4^3(m^2 - 1)}{32c_6^2m^2}$, $c_2 = \frac{c_4^2(5m^2 - 1)}{16c_6m^2}$, $c_6 > 0$, then the coupled system Eqs. (2) and (3) has the Jacobi elliptic functions solutions:

$$u(x, t) = \pm\sqrt{\frac{2m^2\rho_2}{(m^2 + 1)\rho_3}}\ \mathrm{sn}\left[\sqrt{-\frac{\rho_2}{(m^2 + 1)}}\ (x - vt),\ m\right]e^{i(-kx+\omega t+\theta)}, \qquad v(x, t) = \lambda_1u(x, t) \tag{34}$$

or

$$u(x, t) = \pm\sqrt{\frac{2\rho_2}{(m^2 + 1)\rho_3}}\ \mathrm{ns}\left[\sqrt{-\frac{\rho_2}{(m^2 + 1)}}\ (x - vt),\ m\right]e^{i(-kx+\omega t+\theta)}, \qquad v(x, t) = \lambda_1u(x, t) \tag{35}$$

provided $\rho_2 < 0$ and $\rho_3 < 0$. In particular, if $m \to 1^-$, then we have the dark soliton solutions:

$$u(x, t) = \pm\sqrt{\frac{\rho_2}{\rho_3}}\ \tanh\left[\sqrt{-\frac{\rho_2}{2}}\ (x - vt)\right]e^{i(-kx+\omega t+\theta)}, \qquad v(x, t) = \lambda_1u(x, t) \tag{36}$$

and the singular soliton solutions:

$$u(x, t) = \pm\sqrt{\frac{\rho_2}{\rho_3}}\ \coth\left[\sqrt{-\frac{\rho_2}{2}}\ (x - vt)\right]e^{i(-kx+\omega t+\theta)}, \qquad v(x, t) = \lambda_1u(x, t) \tag{37}$$

while if $m \to 0^+$, then we have the periodic solutions:

$$u(x, t) = \pm\sqrt{\frac{2\rho_2}{\rho_3}}\ \csc[\sqrt{-\rho_2}\ (x - vt)]e^{i(-kx+\omega t+\theta)}, \qquad v(x, t) = \lambda_1u(x, t). \tag{38}$$

Family-2: If $c_0 = \frac{c_4^3(1-m^2)}{32c_6^2}$, $c_2 = \frac{c_4^2(5-m^2)}{16c_6}$, $c_6 > 0$, then the coupled system Eqs. (2) and (3) has the same Jacobi elliptic functions solutions (34)–(35).

Family-3: If $c_0 = \frac{c_4^3}{32m^2c_6^2}$, $c_2 = \frac{c_4^2(4m^2+1)}{16c_6m^2}$, $c_6 < 0$, then the coupled system Eqs. (2) and (3) has the Jacobi elliptic functions solutions:

$$u(x,t) = \pm\sqrt{\frac{2m^2\rho_2}{(2m^2-1)\rho_3}}\,\mathrm{cn}\left[\sqrt{\frac{\rho_2}{(2m^2-1)}}\;(x-vt),\, m\right]\mathrm{e}^{i(-kx+\omega t+\theta)},\qquad v(x,t) = \lambda_1 u(x,t) \tag{39}$$

or

$$u(x,t) = \pm\sqrt{\frac{2m^2(1-m^2)\rho_2}{(2m^2-1)\rho_3}}\,\mathrm{sd}\left[\sqrt{\frac{\rho_2}{(2m^2-1)}}\;(x-vt),\, m\right]\mathrm{e}^{i(-kx+\omega t+\theta)},\qquad v(x,t) = \lambda_1 u(x,t) \tag{40}$$

provided $\rho_2(2m^2-1) > 0$ and $\rho_3 > 0$. In particular, if $m \to 1^-$, then we have the bright soliton solutions:

$$u(x,t) = \pm\sqrt{\frac{2\rho_2}{\rho_3}}\,\mathrm{sech}[\sqrt{\rho_2}\;(x-vt)]\mathrm{e}^{i(-kx+\omega t+\theta)},\qquad v(x,t) = \lambda_1 u(x,t) \tag{41}$$

provided $\rho_2 > 0$ and $\rho_3 > 0$.

Family-4: If $c_0 = \frac{c_4^3m^2}{32c_6^2(m^2-1)}$, $c_2 = \frac{c_4^2(5m^2-4)}{16c_6(m^2-1)}$, $c_6 < 0$, then the coupled system Eqs. (2) and (3) has the Jacobi elliptic functions solutions:

$$u(x,t) = \pm\sqrt{\frac{2\rho_2}{(2-m^2)\rho_3}}\,\mathrm{dn}\left[\sqrt{\frac{\rho_2}{(2-m^2)}}\;(x-vt),\, m\right]\mathrm{e}^{i(-kx+\omega t+\theta)},\qquad v(x,t) = \lambda_1 u(x,t) \tag{42}$$

or

$$u(x,t) = \pm\sqrt{\frac{2(1-m^2)\rho_2}{(2-m^2)\rho_3}}\,\mathrm{nd}\left[\sqrt{\frac{\rho_2}{(2-m^2)}}\;(x-vt),\, m\right]\mathrm{e}^{i(-kx+\omega t+\theta)},\qquad v(x,t) = \lambda_1 u(x,t) \tag{43}$$

provided $\rho_2 > 0$ and $\rho_3 > 0$.In particular, if $m \to 1^-$, then we have the same bright soliton solutions Eq. (41).

Family-5: If $c_0 = \frac{c_4^3}{32c_6^2(1-m^2)}$, $c_2 = \frac{c_4^2(4m^2-5)}{16c_6(m^2-1)}$, $c_6 > 0$, then the coupled system Eqs. (2) and (3) has the Jacobi elliptic functions solutions:

$$u(x,t) = \pm\sqrt{-\frac{2(1-m^2)\rho_2}{(2m^2-1)\rho_3}}\,\mathrm{nc}\left[\sqrt{\frac{\rho_2}{(2m^2-1)}}\;(x-vt),\, m\right]\mathrm{e}^{i(-kx+\omega t+\theta)},\qquad v(x,t) = \lambda_1 u(x,t) \tag{44}$$

or

$$u(x,t) = \pm\sqrt{-\frac{2\rho_2}{(2m^2-1)\rho_3}}\,\mathrm{ds}\left[\sqrt{\frac{\rho_2}{(2m^2-1)}}\;(x-vt),\, m\right]\mathrm{e}^{i(-kx+\omega t+\theta)},\qquad v(x,t) = \lambda_1 u(x,t) \tag{45}$$

provided $(2m^2-1)\rho_2 > 0$ and $\rho_3 < 0$. If $m \to 1^-$, then we have the singular soliton solutions:

$$u(x,t) = \pm\sqrt{-\frac{2\rho_2}{\rho_3}}\,\mathrm{csch}[\sqrt{\rho_2}\;(x-vt)]\mathrm{e}^{i(-kx+\omega t+\theta)},\qquad v(x,t) = \lambda_1 u(x,t) \tag{46}$$

provided $\rho_2 > 0$ and $\rho_3 < 0$, where if $m \to 0^+$, then we have the same periodic solutions Eq. (38) as well as the periodic solutions:

$$u(x,t) = \pm\sqrt{\frac{2\rho_2}{\rho_3}}\,\sec[\sqrt{-\rho_2}\;(x-vt)]\mathrm{e}^{i(-kx+\omega t+\theta)},\qquad v(x,t) = \lambda_1 u(x,t) \tag{47}$$

provided $\rho_2 < 0$ and $\rho_3 < 0$.

Family-6: If $c_0 = \frac{c_4^3}{32c_6^2(1-m^2)}$, $c_2 = \frac{c_4^2(4m^2-5)}{16c_6(m^2-1)}$, $c_6 > 0$, then the coupled system (2) and (3) has the same Jacobi elliptic functions solutions (42) and (43).

3. Mathematical analysis: Case-II ($n = 1$)

To this aim, we make the same transforms Eqs. (8) and (9). Substituting Eqs. (8) and (9) into the system Eqs. (4) and (5), separating the real and the imaginary parts of the system Eqs. (4) and (5), we have:

$$a_1\varphi_2'' - (\omega - k\alpha_1)\varphi_1 + (\beta_1 - a_1k^2)\varphi_2 + \frac{f_1\varphi_1}{b_1\varphi_1^4 + c_1\varphi_1^2\varphi_2^2 + d_1\varphi_2^4} + (\xi_1\varphi_1^2 + \eta_1\varphi_2^2)\varphi_1 + (\theta_1\varphi_1^4 + \gamma_1\varphi_1^2\varphi_2^2 + \delta_1\varphi_2^4)\varphi_1 = 0, \tag{48}$$

$$a_2\varphi_1'' - (\omega - k\alpha_2)\varphi_2 + (\beta_2 - a_2k^2)\varphi_1 + \frac{f_2\varphi_2}{b_2\varphi_2^4 + c_2\varphi_2^2\varphi_1^2 + d_2\varphi_1^4} + (\xi_2\varphi_2^2 + \eta_2\varphi_1^2)\varphi_2 + (\theta_2\varphi_2^4 + \gamma_2\varphi_2^2\varphi_1^2 + \delta_2\varphi_1^4)\varphi_2 = 0, \tag{49}$$

and

$$(\upsilon - \alpha_1)\varphi_1' + 2a_1k\varphi_2' = 0, \tag{50}$$

$$2a_2k\varphi_1' + (\upsilon - \alpha_2)\varphi_2' = 0. \tag{51}$$

Setting

$$\varphi_2(\xi) = \lambda_2\varphi_1(\xi), \tag{52}$$

where λ_2 is a non zero constant, such that $\lambda_2 \neq 1$. Consequently, Eqs. (48)–(51) reduce to

$$a_1\lambda_2\varphi_1'' - [\omega - k\alpha_1 - \lambda_2(\beta_1 - a_1k^2)]\varphi_1 + \frac{f_1}{(b_1 + \lambda_2^2c_1 + \lambda_2^4d_1)\varphi_1^3} + (\xi_1 + \lambda_2^2\eta_1)\varphi_1^3 + (\theta_1 + \lambda_2^2\gamma_1 + \lambda_2^4\delta_1)\varphi_1^5 = 0, \tag{53}$$

$$a_2\varphi_1'' - [\lambda_2(\omega - k\alpha_2) - (\beta_2 - a_2k^2)]\varphi_1 + \frac{\lambda_2f_2}{(\lambda_2^4b_2 + c_2\lambda_2^2 + d_2)\varphi_1^3} + \lambda_2(\lambda_2^2\xi_2 + \eta_2)\varphi_1^3 + \lambda_2(\lambda_2^4\theta_2 + \lambda_2^2\gamma_2 + \delta_2)\varphi_1^5 = 0, \tag{54}$$

and

$$[\upsilon - \alpha_1 + 2a_1k\lambda_2]\varphi_1' = 0, \tag{55}$$

$$[2a_2k + \lambda_2(\upsilon - \alpha_2)]\varphi_1' = 0. \tag{56}$$

From Eqs. (55) and (56), one can obtain the velocity of the soliton as

$$\upsilon = \alpha_1 - 2a_1k\lambda_2, \tag{57}$$

and

$$\upsilon = \frac{\alpha_2\lambda_2 - 2a_2k}{\lambda_2}. \tag{58}$$

From Eqs. (57) and (58), we have the constraint condition:

$$2a_1k\lambda_2^2 + (\alpha_2 - \alpha_1)\lambda_2 - 2a_2k = 0. \tag{59}$$

From Eq. (59), we have

$$\lambda_2 = \frac{-(\alpha_2 - \alpha_1) \pm \sqrt{(\alpha_2 - \alpha_1)^2 + 16a_1a_2k^2}}{4a_1k}, \tag{60}$$

provided $a_1a_2 > 0$. Eqs. (53) and (54) have the same form under the constraint conditions:

$$a_1\lambda_2 = a_2, \tag{61}$$

$$(\xi_1 + \lambda_2^2\eta_1) = \lambda_2(\lambda_2^2\xi_2 + \eta_2), \tag{62}$$

$$(\theta_1 + \lambda_2^2\gamma_1 + \lambda_2^4\delta_1) = \lambda_2(\lambda_2^4\theta_2 + \lambda_2^2\gamma_2 + \delta_2). \tag{63}$$

$$f_1(\lambda_2^4b_2 + c_2\lambda_2^2 + d_2) = \lambda_2f_2(b_1 + \lambda_2^2c_1 + \lambda_2^4d_1), \tag{64}$$

$$[\omega - k\alpha_1 - \lambda_2(\beta_1 - a_1k^2)] = [\lambda_2(\omega - k\alpha_2) - (\beta_2 - a_2k^2)]. \tag{65}$$

Eq. (53) can be rewritten in the form:

$$\varphi_1^3\varphi_1'' + \mu_1 - \mu_2\varphi_1^4 + \mu_3\varphi_1^6 + \mu_4\varphi_1^8 = 0, \tag{66}$$

where

$$\mu_1 = \frac{f_1}{a_1\lambda_2(b_1 + \lambda_2^2 c_1 + \lambda_2^4 d_1)}, \mu_2 = \frac{[\omega - k\alpha_1 - \lambda_2(\beta_1 - a_1 k^2)]}{a_1\lambda_2},$$
$$\mu_3 = \frac{(\xi_1 + \lambda_2^2\eta_1)}{a_1\lambda_2}, \mu_4 = \frac{(\theta_1 + \lambda_2^2\gamma_1 + \lambda_2^4\delta_1)}{a_1\lambda_2}, \tag{67}$$

provided $a_1\lambda_2(b_1 + \lambda_2^2 c_1 + \lambda_2^4 d_1) \neq 0$. Balancing $\varphi_1^3(\xi)\varphi_1''(\xi)$ with $\varphi_1^8(\xi)$ in Eq. (66) yields $N = \frac{1}{2}$. Setting

$$\varphi_1(\xi) = H^{\frac{1}{2}}(\xi), \tag{68}$$

where $H(\xi)$ is a new function of ξ, we get the new equation

$$H'^2 - 2HH'' - 4\mu_1 + 4\mu_2 H^2 - 4\mu_3 H^3 - 4\mu_4 H^4 = 0, \tag{69}$$

Balancing [44–58] $H(\xi)H''(\xi)$ with $H^4(\xi)$ in Eq. (69) yields $N = 1$. In the next subsection, we solve Eq. (69) using the proposed method.

3.1. Extended auxiliary equation

According to this method, we assume that Eq. (69) has the formal solution:

$$H(\xi) = A_0 + A_1 F(\xi) + A_2 F^2(\xi), \tag{70}$$

where A_0, A_1 and A_2 are constants to be determined, such that $A_2 \neq 0$, while the function $F(\xi)$ satisfies the first order Eq. (30). Substituting (70) along with Eq. (30) into Eq. (69), collecting the coefficients of each power $F^i(\xi)(F'(\xi))^j$, $(i = 0, 1, 2, .,8,\ \ j = 0, 1)$, and setting these coefficients to zero, we have a set of algebraic equations which can be solved by the aid of Maple to obtain the following results:

$$c_0 = -\frac{A_0(A_0^2\mu_4 + A_0\mu_3 - \mu_2)}{A_2}, c_2 = -2\mu_4 A_0^2 - \frac{3}{2}A_0\mu_3 + \mu_2, c_4 = -\frac{4}{3}A_0 A_2\mu_4 - \frac{1}{2}A_2\mu_3,$$
$$c_6 = -\frac{1}{3}A_2^2\mu_4, A_0 = A_0, A_1 = 0, A_2 = A_2, \mu_1 = 0, \mu_2 = \mu_2, \mu_3 = \mu_3, \mu_4 = \mu_4. \tag{71}$$

From Eqs. (31), (68), (70) and (71), then we have the solutions:

$$\varphi_1(\xi) = \left\{-\frac{3\mu_3}{8\mu_4} \pm \left[A_0 + \frac{3\mu_3}{8\mu_4}\right] f(\xi)\right\}^{\frac{1}{2}}. \tag{72}$$

The coupled system (4) and ((5) has the following families of solutions:

Family-1: If $c_0 = \frac{c_4^3(m^2-1)}{32c_6^2 m^2}$, $c_2 = \frac{c_4^2(5m^2-1)}{16c_6 m^2}$, $c_6 > 0$, then the coupled system (4) and ((5) has the Jacobi elliptic functions solutions:

$$u(x,t) = \pm\left\{-\frac{3\mu_3}{8\mu_4}\left(1 \pm \text{sn}\left[\sqrt{-\frac{3\mu_3^2}{16m^2\mu_4}}\ (x - \upsilon t), m\right]\right)\right\}^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)}, \qquad v(x,t) = \lambda_2 u(x,t) \tag{73}$$

or

$$u(x,t) = \pm\left\{-\frac{3\mu_3}{8\mu_4}\left(1 \pm \frac{1}{m}\ \text{ns}\left[\sqrt{-\frac{3\mu_3^2}{16m^2\mu_4}}\ (x - \upsilon t), m\right]\right)\right\}^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)}, \qquad v(x,t) = \lambda_2 u(x,t) \tag{74}$$

provided $\mu_4 < 0$ and $\mu_3 > 0$. In particular, if $m \to 1^-$, then we have the dark soliton solutions:

$$u(x,t) = \pm\left\{-\frac{3\mu_3}{8\mu_4}\left(1 \pm \tanh\left[\sqrt{-\frac{3\mu_3^2}{16\mu_4}}\ (x - \upsilon t)\right]\right)\right\}^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)}, \qquad v(x,t) = \lambda_2 u(x,t) \tag{75}$$

and the singular soliton solutions:

$$u(x,t) = \pm\left\{-\frac{3\mu_3}{8\mu_4}\left(1 \pm \coth\left[\sqrt{-\frac{3\mu_3^2}{16\mu_4}}\ (x - \upsilon t)\right]\right)\right\}^{\frac{1}{2}} e^{i(-kx+\omega t+\theta)}, \qquad v(x,t) = \lambda_2 u(x,t). \tag{76}$$

Family-2: If $c_0 = \frac{c_4^3(1-m^2)}{32c_6^2}$, $c_2 = \frac{c_4^2(5-m^2)}{16c_6}$, $c_6 > 0$, then the coupled system Eqs. (4) and (5) has the Jacobi elliptic functions solutions:

$$u(x,t)=\pm\left\{-\frac{3\mu_3}{8\mu_4}\left(1\pm m\mathrm{sn}\left[\sqrt{-\frac{3\mu_3^2}{16\mu_4}}\ (x-vt),\ m\right]\right)\right\}^{\frac{1}{2}}\ e^{i(-kx+\omega t+\theta)},\qquad v(x,t)=\lambda_2 u(x,t) \tag{77}$$

or

$$u(x,t)=\pm\left\{-\frac{3\mu_3}{8\mu_4}\left(1\pm \mathrm{ns}\left[\sqrt{-\frac{3\mu_3^2}{16\mu_4}}\ (x-vt),\ m\right]\right)\right\}^{\frac{1}{2}}\ e^{i(-kx+\omega t+\theta)},\qquad v(x,t)=\lambda_2 u(x,t) \tag{78}$$

provided $\mu_4 < 0$ and $\mu_3 > 0$. In particular, if $m \to 1^-$, then we have the same dark soliton solutions Eq. (75) and the same singular soliton solutions (76), while if $m \to 0^+$, then we have the periodic solution:

$$u(x,t)=\pm\left\{-\frac{3\mu_3}{8\mu_4}\left(1\pm \csc\left[\sqrt{-\frac{3\mu_3^2}{16\mu_4}}\ (x-vt)\right]\right)\right\}^{\frac{1}{2}}\ e^{i(-kx+\omega t+\theta)},\quad v(x,t)=\lambda_2 u(x,t). \tag{79}$$

Family 3. If $c_0=\frac{c_4^3}{32m^2c_6^2}$, $c_2=\frac{c_4^2(4m^2+1)}{16c_6m^2}$, $c_6<0$, then the coupled system Eqs. (4) and (5) has the Jacobi elliptic functions solutions:

$$u(x,t)=\pm\left\{-\frac{3\mu_3}{8\mu_4}\left(1\pm \mathrm{cn}\left[\sqrt{\frac{3\mu_3^2}{16m^2\mu_4}}\ (x-vt),\ m\right]\right)\right\}^{\frac{1}{2}}\ e^{i(-kx+\omega t+\theta)},\qquad v(x,t)=\lambda_2 u(x,t) \tag{80}$$

or

$$u(x,t)=\pm\left\{-\frac{3\mu_3}{8\mu_4}\left(1\pm \sqrt{1-m^2}\,\mathrm{sd}\left[\sqrt{\frac{3\mu_3^2}{16m^2\mu_4}}\ (x-vt),\ m\right]\right)\right\}^{\frac{1}{2}}\ e^{i(-kx+\omega t+\theta)},\qquad v(x,t)=\lambda_2 u(x,t) \tag{81}$$

provided $\mu_4 > 0$ and $\mu_3 < 0$. In particular, if $m \to 1^-$, then we have the bright soliton solutions:

$$u(x,t)=\pm\left\{-\frac{3\mu_3}{8\mu_4}\left(1\pm \mathrm{sech}\left[\sqrt{\frac{3\mu_3^2}{16\mu_4}}\ (x-vt)\right]\right)\right\}^{\frac{1}{2}}\ e^{i(-kx+\omega t+\theta)},\qquad v(x,t)=\lambda_2 u(x,t). \tag{82}$$

Family 4. If $c_0=\frac{c_4^3m^2}{32c_6^2(m^2-1)}$, $c_2=\frac{c_4^2(5m^2-4)}{16c_6(m^2-1)}$, $c_6<0$, then the coupled system Eqs. (4) and (5) has the Jacobi elliptic functions solutions:

$$u(x,t)=\pm\left\{-\frac{3\mu_3}{8\mu_4}\left(1\pm \frac{1}{\sqrt{1-m^2}}\mathrm{dn}\left[\sqrt{\frac{3\mu_3^2}{16(1-m^2)\mu_4}}\ (x-vt),\ m\right]\right)\right\}^{\frac{1}{2}}\ e^{i(-kx+\omega t+\theta)},\qquad v(x,t)=\lambda_2 u(x,t) \tag{83}$$

or

$$u(x,t)=\pm\left\{-\frac{3\mu_3}{8\mu_4}\left(1\pm \mathrm{nd}\left[\sqrt{\frac{3\mu_3^2}{16(1-m^2)\mu_4}}\ (x-vt),\ m\right]\right)\right\}^{\frac{1}{2}}\ e^{i(-kx+\omega t+\theta)},\qquad v(x,t)=\lambda_2 u(x,t) \tag{84}$$

provided $\mu_4 > 0$ and $\mu_3 < 0$.

Family-5: If $c_0=\frac{c_4^3}{32c_6^2(1-m^2)}$, $c_2=\frac{c_4^2(4m^2-5)}{16c_6(m^2-1)}$, $c_6>0$, then the coupled system Eqs. (4) and (5) has the Jacobi elliptic functions solutions:

$$u(x,t)=\pm\left\{-\frac{3\mu_3}{8\mu_4}\left(1\pm \mathrm{nc}\left[\sqrt{-\frac{3\mu_3^2}{16(1-m^2)\mu_4}}\ (x-vt),\ m\right]\right)\right\}^{\frac{1}{2}}\ e^{i(-kx+\omega t+\theta)},\qquad v(x,t)=\lambda_2 u(x,t) \tag{85}$$

or

$$u(x,t)=\pm\left\{-\frac{3\mu_3}{8\mu_4}\left(1\pm \frac{1}{\sqrt{1-m^2}}\mathrm{ds}\left[\sqrt{-\frac{3\mu_3^2}{16(1-m^2)\mu_4}}\ (x-vt),\ m\right]\right)\right\}^{\frac{1}{2}}\ e^{i(-kx+\omega t+\theta)},\qquad v(x,t)=\lambda_2 u(x,t) \tag{86}$$

provided $\mu_4 < 0$ and $\mu_3 > 0$. If $m \to 0^+$, then we have the same periodic solutions Eq. (79) as well as the periodic solutions:

$$u(x,t)=\pm\left\{-\frac{3\mu_3}{8\mu_4}\left(1\pm\sec\left[\sqrt{-\frac{3\mu_3^2}{16\mu_4}}\,(x-vt)\right]\right)\right\}^{\frac{1}{2}}e^{i(-kx+\omega t+\theta)},\qquad v(x,t)=\lambda_2 u(x,t). \tag{87}$$

Family-6: If $c_0=\frac{c_4^3}{32c_6^2(1-m^2)}$, $c_2=\frac{c_4^2(4m^2-5)}{16c_6(m^2-1)}$, $c_6>0$, then the coupled system Eqs. (4) and (5) has the Jacobi elliptic functions solutions:

$$u(x,t)=\pm\left\{-\frac{3\mu_3}{8\mu_4}\left(1\pm\mathrm{dn}\left[\sqrt{\frac{3\mu_3^2}{16\mu_4}}\,(x-vt),\,m\right]\right)\right\}^{\frac{1}{2}}e^{i(-kx+\omega t+\theta)},\qquad v(x,t)=\lambda_2 u(x,t) \tag{88}$$

or

$$u(x,t)=\pm\left\{-\frac{3\mu_3}{8\mu_4}\left(1\pm\sqrt{1-m^2}\,\mathrm{nd}\left[\sqrt{\frac{3\mu_3^2}{16\mu_4}}\,(x-vt),\,m\right]\right)\right\}^{\frac{1}{2}}e^{i(-kx+\omega t+\theta)},\qquad v(x,t)=\lambda_2 u(x,t) \tag{89}$$

provided $\mu_4>0$ and $\mu_3<0$. In particular, if $m\to 1^-$, then we have the same bright soliton solutions Eq. (82).

4. Mathematical analysis: Case-III ($n=2$)

To this aim, we make the same transforms Eqs. (8) and (9). Substituting Eqs. (8) and (9) into the system Eqs. (6) and (7), separating the real and the imaginary parts of the system Eqs. (6) and (7), we have:

$$a_1\varphi_2''-(\omega-k\alpha_1)\varphi_1+(\beta_1-a_1k^2)\varphi_2+\frac{f_1\varphi_1}{b_1\varphi_1^6+c_1\varphi_1^4\varphi_2^2+d_1\varphi_1^2\varphi_2^4+e_1\varphi_1^6}+(\theta_1\varphi_1^4+\gamma_1\varphi_1^2\varphi_2^2+\delta_1\varphi_2^4)\varphi_1 \\ +(\xi_1\varphi_1^6+\sigma_1\varphi_1^4\varphi_2^2+r_1\varphi_1^2\varphi_2^4+\eta_1\varphi_2^6)\varphi_1=0, \tag{90}$$

$$a_2\varphi_1''-(\omega-k\alpha_2)\varphi_2+(\beta_2-a_2k^2)\varphi_1+\frac{f_2\varphi_2}{b_2\varphi_2^6+c_2\varphi_2^4\varphi_1^2+d_2\varphi_2^2\varphi_1^4+e_2\varphi_2^6}+(\theta_2\varphi_2^4+\gamma_2\varphi_2^2\varphi_1^2+\delta_2\varphi_1^4)\varphi_2 \\ +(\xi_2\varphi_2^6+\sigma_2\varphi_2^4\varphi_1^2+r_2\varphi_2^2\varphi_1^4+\eta_2\varphi_1^6)\varphi_2=0, \tag{91}$$

and

$$(v-\alpha_1)\varphi_1'+2a_1k\varphi_2'=0, \tag{92}$$

$$2a_2k\varphi_1'+(v-\alpha_2)\varphi_2'=0, \tag{93}$$

Setting

$$\varphi_2(\xi)=\lambda_3\varphi_1(\xi), \tag{94}$$

where λ_3 is a non zero constant, such that $\lambda_3\neq 1$. Consequently, Eqs. (90)–(93) reduce to

$$a_1\lambda_3\varphi_1''-[\omega-k\alpha_1-\lambda_3(\beta_1-a_1k^2)]\varphi_1+\frac{f_1}{(b_1+c_1\lambda_3^2+d_1\lambda_3^4+e_1)\varphi_1^5}+(\theta_1+\gamma_1\lambda_3^2+\lambda_3^4\delta_1)\varphi_1^5 \\ +(\xi_1+\lambda_3^2\sigma_1+\lambda_3^4r_1+\lambda_3^6\eta_1)\varphi_1^7=0, \tag{95}$$

$$a_2\varphi_1''-[\lambda_3(\omega-k\alpha_2)-(\beta_2-a_2k^2)]\varphi_1+\frac{\lambda_3f_2}{(b_2\lambda_3^6+\lambda_3^4c_2+\lambda_3^2d_2+\lambda_3^6e_2)\varphi_1^5}+\lambda_3(\lambda_3^4\theta_2+\lambda_3^2\gamma_2+\delta_2)\varphi_1^5 \\ +\lambda_3(\lambda_3^6\xi_2+\lambda_3^4\sigma_2+\lambda_3^2r_2+\eta_2)\varphi_1^7=0, \tag{96}$$

and

$$(v-\alpha_1)\varphi_1'+2a_1k\lambda_3\varphi_1'=0, \tag{97}$$

$$2a_2k\varphi_1'+\lambda_3(v-\alpha_2)\varphi_1'=0. \tag{98}$$

From Eqs. (97) and (98), one can obtain the velocity of the soliton as

$$v=\alpha_1-2a_1k\lambda_3, \tag{99}$$

and

$$v=\frac{\alpha_2\lambda_3-2a_2k}{\lambda_3}. \tag{100}$$

From Eqs. (99) and (100), we have the constraint condition:

$$2a_1k\lambda_3^2+(\alpha_2-\alpha_1)\lambda_3-2a_2k=0. \tag{101}$$

From Eq. (101), we have

$$\lambda_3=\frac{-(\alpha_2-\alpha_1)\pm\sqrt{(\alpha_2-\alpha_1)^2+16a_1a_2k^2}}{4a_1k}, \tag{102}$$

provided $a_1a_2>0$. Eqs. (95) and (96) have the same form under the constraint conditions:

$$a_1\lambda_3=a_2, \tag{103}$$

$$(\theta_1+\gamma_1\lambda_3^2+\lambda_3^4\delta_1)=\lambda_3(\lambda_3^4\theta_2+\lambda_3^2\gamma_2+\delta_2). \tag{104}$$

$$[\omega-k\alpha_1-\lambda_3(\beta_1-a_1k^2)]=[\lambda_3(\omega-k\alpha_2)-(\beta_2-a_2k^2)]. \tag{105}$$

$$(\xi_1+\lambda_3^2\sigma_1+\lambda_3^4r_1+\lambda_3^6\eta_1)=\lambda_3(\lambda_3^6\xi_2+\lambda_3^4\sigma_2+\lambda_3^2r_2+\eta_2), \tag{106}$$

$$f_1(b_2\lambda_3^6+\lambda_3^4c_2+\lambda_3^2d_2+\lambda_3^6e_2)=\lambda_3f_2(b_1+c_1\lambda_3^2+d_1\lambda_3^4+e_1). \tag{107}$$

Eq. (95) can be rewritten in the form:

$$\varphi_1^5\varphi_1''+K_1-K_2\varphi_1^6+K_3\varphi_1^{10}+K_4\varphi_1^{12}=0, \tag{108}$$

where

$$K_1=\frac{f_1}{a_1\lambda_3(b_1+c_1\lambda_3^2+d_1\lambda_3^4+e_1)},\ K_2=\frac{[\omega-k\alpha_1-\lambda_3(\beta_1-a_1k^2)]}{a_1\lambda_3},$$
$$K_3=\frac{(\theta_1+\gamma_1\lambda_3^2+\lambda_3^4\delta_1)}{a_1\lambda_3},\ K_4=\frac{(\xi_1+\lambda_3^2\sigma_1+\lambda_3^4r_1+\lambda_3^6\eta_1)}{a_1\lambda_3}, \tag{109}$$

provided $a_1\lambda_3(b_1+c_1\lambda_3^2+d_1\lambda_3^4+e_1)\neq0$. Balancing $\varphi_1^5(\xi)\varphi_1''(\xi)$ with $\varphi_1^{12}(\xi)$ in Eq. (108) yields $N=\frac{1}{3}$. Setting

$$\varphi_1(\xi)=G^{\frac{1}{3}}(\xi), \tag{110}$$

where $G(\xi)$ is a new function of ξ, we get the new equation

$$2G'^2-3GG''-9K_1+9K_2G^2-9K_3G^{\frac{10}{3}}-9K_4G^4=0. \tag{111}$$

For integrability, one must select $K_3=0$. This leads to the modification of the model of study as:

$$2G'^2-3GG''-9K_1+9K_2G^2-9K_4G^4=0. \tag{112}$$

Balancing [44–58] $G(\xi)G''(\xi)$with $G^4(\xi)$ in Eq. (112) yields $N=1$. In the next subsection, we solve Eq. (112) using the proposed method.

4.1. Extended auxiliary equation

According to this method, we assume that Eq. (112) has the formal solution:

$$G(\xi)=A_0+A_1F(\xi)+A_2F^2(\xi), \tag{113}$$

where A_0, A_1 and A_2 are constants to be determined, such that $A_2\neq0$, while the function $F(\xi)$ satisfies the first order Eq. (30). Substituting Eq. (113) along with Eq. (30) into Eq. (112), collecting the coefficients of each power $F^i(\xi)(F'(\xi))^j$, $(i=0,1,2,..,8,\ \ j=0,1)$, and setting these coefficients to zero, we have a set of algebraic equations which can be solved by the aid of Maple to obtain the following results:

$$c_0=-\frac{9A_0(4A_0^2K_4-5K_2)}{40A_2},\ c_2=-\frac{27}{20}A_0^2K_4+\frac{9}{16}K_2,\ c_4=-\frac{9}{10}A_0A_2K_4,\ c_6=-\frac{9}{40}A_2^2K_4,$$
$$A_0=A_0,\ A_1=0,\ A_2=A_2,\ K_1=\frac{1}{5}A_0^4K_4-\frac{1}{2}A_0^2K_2,\ K_2=K_2,\ K_4=K_4. \tag{114}$$

From Eqs. (31), (110), (113), and (114), then we have the solutions:

$$\varphi_1(\xi)=\{\pm A_0\ f(\xi)\}^{\frac{1}{3}}. \tag{115}$$

The coupled system Eqs. (6) and (7) has the following families of solutions:

Family-1: If $c_0=\dfrac{c_4^3(m^2-1)}{32c_6^2m^2}$, $c_2=\dfrac{c_4^2(5m^2-1)}{16c_6m^2}$, $c_6>0$, then the coupled system Eqs. (6) and (7) has the Jacobi elliptic functions solutions:

$$u(x,t)=\pm\left\{\sqrt{\frac{5m^2K_2}{2(m^2+1)K_4}}\,\mathrm{sn}\left[\sqrt{-\frac{9K_2}{4(m^2+1)}}\,(x-vt),\,m\right]\right\}^{\frac{1}{3}}e^{i(-kx+\omega t+\theta)},\qquad v(x,t)=\lambda_3u(x,t)\tag{116}$$

or

$$u(x,t)=\pm\left\{\sqrt{\frac{5K_2}{2(m^2+1)K_4}}\,\mathrm{ns}\left[\sqrt{-\frac{9K_2}{4(m^2+1)}}\,(x-vt),\,m\right]\right\}^{\frac{1}{3}}e^{i(-kx+\omega t+\theta)},\qquad v(x,t)=\lambda_3u(x,t)\tag{117}$$

provided $K_2<0$ and $K_4<0$. In particular, if $m\to 1^-$, then we have the dark soliton solutions:

$$u(x,t)=\pm\left\{\sqrt{\frac{5K_2}{4K_4}}\tanh\left[\sqrt{-\frac{9K_2}{8}}\,(x-vt)\right]\right\}^{\frac{1}{3}}e^{i(-kx+\omega t+\theta)},\qquad v(x,t)=\lambda_3u(x,t)\tag{118}$$

and the singular soliton solutions:

$$u(x,t)=\pm\left\{\sqrt{\frac{5K_2}{4K_4}}\coth\left[\sqrt{-\frac{9K_2}{8}}\,(x-vt)\right]\right\}^{\frac{1}{3}}e^{i(-kx+\omega t+\theta)},\qquad v(x,t)=\lambda_3u(x,t)\tag{119}$$

while if $m\to 0^+$, then we have the periodic solutions:

$$u(x,t)=\pm\left\{\sqrt{\frac{5K_2}{2K_4}}\csc\left[\sqrt{-\frac{9K_2}{4}}\,(x-vt)\right]\right\}^{\frac{1}{3}}e^{i(-kx+\omega t+\theta)},\qquad v(x,t)=\lambda_3u(x,t).\tag{120}$$

Family-2: If $c_0=\frac{c_4^3(1-m^2)}{32c_6^2}$, $c_2=\frac{c_4^2(5-m^2)}{16c_6}$, $c_6>0$, then the coupled system Eqs. (6) and (7) has the same Jacobi elliptic functions solutions Eqs. (116) and (117).

Family-3: If $c_0=\frac{c_4^3}{32m^2c_6^2}$, $c_2=\frac{c_4^2(4m^2+1)}{16c_6m^2}$, $c_6<0$, then the coupled system Eqs. (6) and (7) has the Jacobi elliptic functions solutions:

$$u(x,t)=\pm\left\{\sqrt{\frac{5m^2K_2}{2(2m^2-1)K_4}}\,\mathrm{cn}\left[\sqrt{\frac{9K_2}{4(2m^2-1)}}\,(x-vt),\,m\right]\right\}^{\frac{1}{3}}e^{i(-kx+\omega t+\theta)},\qquad v(x,t)=\lambda_3u(x,t)\tag{121}$$

or

$$u(x,t)=\pm\left\{\sqrt{\frac{5m^2(1-m^2)K_2}{2(2m^2-1)K_4}}\,\mathrm{sd}\left[\sqrt{\frac{9K_2}{4(2m^2-1)}}\,(x-vt),\,m\right]\right\}^{\frac{1}{3}}e^{i(-kx+\omega t+\theta)},\qquad v(x,t)=\lambda_3u(x,t)\tag{122}$$

provided $(2m^2-1)K_2>0$ and $K_4>0$. In particular, if $m\to 1^-$, then we have the bright soliton solutions:

$$u(x,t)=\pm\left\{\sqrt{\frac{5K_2}{2K_4}}\,\mathrm{sech}\left[\sqrt{\frac{9K_2}{4}}\,(x-vt)\right]\right\}^{\frac{1}{3}}e^{i(-kx+\omega t+\theta)},\qquad v(x,t)=\lambda_3u(x,t)\tag{123}$$

provided $K_2>0$ and $K_4>0$.

Family-4: If $c_0=\frac{c_4^3m^2}{32c_6^2(m^2-1)}$, $c_2=\frac{c_4^2(5m^2-4)}{16c_6(m^2-1)}$, $c_6<0$, then the coupled system Eqs. (6) and (7) has the Jacobi elliptic functions solutions:

$$u(x,t)=\pm\left\{\sqrt{-\frac{5K_2}{2(m^2-2)K_4}}\,\mathrm{dn}\left[\sqrt{-\frac{9K_2}{4(m^2-2)}}\,(x-vt),\,m\right]\right\}^{\frac{1}{3}}e^{i(-kx+\omega t+\theta)},\qquad v(x,t)=\lambda_3u(x,t)\tag{124}$$

or

$$u(x,t)=\pm\left\{\sqrt{-\frac{5(1-m^2)K_2}{2(m^2-2)K_4}}\,\mathrm{nd}\left[\sqrt{-\frac{9K_2}{4(m^2-2)}}\,(x-vt),\,m\right]\right\}^{\frac{1}{3}}e^{i(-kx+\omega t+\theta)},\qquad v(x,t)=\lambda_3u(x,t)\tag{125}$$

provided $(m^2-2)<0$, $K_2>0$ and $K_4>0$. In particular, if $m\to 1^-$, then we have the same bright soliton solutions Eq. (123).

Family-5: If $c_0=\frac{c_4^3}{32c_6^2(1-m^2)}$, $c_2=\frac{c_4^2(4m^2-5)}{16c_6(m^2-1)}$, $c_6>0$, then the coupled system Eqs. (6) and (7) has the Jacobi elliptic functions solutions:

$$u(x,t)=\pm\left\{\sqrt{-\frac{5(1-m^2)K_2}{2(2m^2-1)K_4}}\,\mathrm{nc}\left[\sqrt{\frac{9K_2}{4(2m^2-1)}}\,(x-vt),\,m\right]\right\}^{\frac{1}{3}}e^{i(-kx+\omega t+\theta)},\qquad v(x,t)=\lambda_3u(x,t)\tag{126}$$

or

$$u(x,t)=\pm\left\{\sqrt{-\frac{5K_2}{2(2m^2-1)K_4}}\,\mathrm{ds}\left[\sqrt{\frac{9K_2}{4(2m^2-1)}}\ (x-vt),\,m\right]\right\}^{\frac{1}{3}}\,e^{i(-kx+\omega t+\theta)},\qquad v(x,t)=\lambda_3 u(x,t) \tag{127}$$

provided $(2m^2-1)K_2>0$ and $K_4<0$. In particular, if $m\to 1^-$, then we have the singular soliton solutions:

$$u(x,t)=\pm\left\{\sqrt{-\frac{5K_2}{2K_4}}\,\mathrm{csch}\left[\sqrt{\frac{9K_2}{4}}\ (x-vt)\right]\right\}^{\frac{1}{3}}\,e^{i(-kx+\omega t+\theta)},\qquad v(x,t)=\lambda_3 u(x,t) \tag{128}$$

provided $K_2>0$ and $K_4<0$, while if $m\to 0^+$, then we have the same periodic solutions Eq. (120) as well as the periodic solutions:

$$u(x,t)=\pm\left\{\sqrt{\frac{5K_2}{2K_4}}\,\sec\left[\sqrt{-\frac{9K_2}{4}}\ (x-vt)\right]\right\}^{\frac{1}{3}}\,e^{i(-kx+\omega t+\theta)},\qquad v(x,t)=\lambda_3 u(x,t) \tag{129}$$

provided $K_2<0$ and $K_4<0$.

Family-6: If $c_0=\frac{c_4^3}{32c_6^2(1-m^2)}$, $c_2=\frac{c_4^2(4m^2-5)}{16c_6(m^2-1)}$, $c_6>0$, then the coupled system Eqs. (6) and (7) has the same Jacobi elliptic functions solutions Eqs. (124) and (125).

5. Numerical simulations

We perform numerical simulations for bright and dark solitions having generalized anti-cubic nonlinearity.

5.1. Bright solitons

The result and the profile of three cases are shown in Figs. 1, 2 and 3 for the data given in the Tables 1, 2 and 3.

5.2. Dark solitons

The result and the profile of three cases are shown in Figs. 4, 5 and 6 for the data given in the Tables 1, 2 and 3.

6. Conclusions

The current paper retrieved optical soliton solutions in fiber BGs that is studied with generalized AC nonlinearity. For arbitrary n, it is not possible to derive the model equations in birefringent fibers or Bragg gratings with a general value of n [31–34] Therefore it is imperative to consider specific value(s) of n as permitted by the stability regime. The extended auxiliary equation is considered since it is suitable and gives the abundance of results which display a wide spectrum of soliton solutions that are listed and enumerated with their respective existence criteria. These solutions satisfy the original equations by using of the Maple. The special cases are

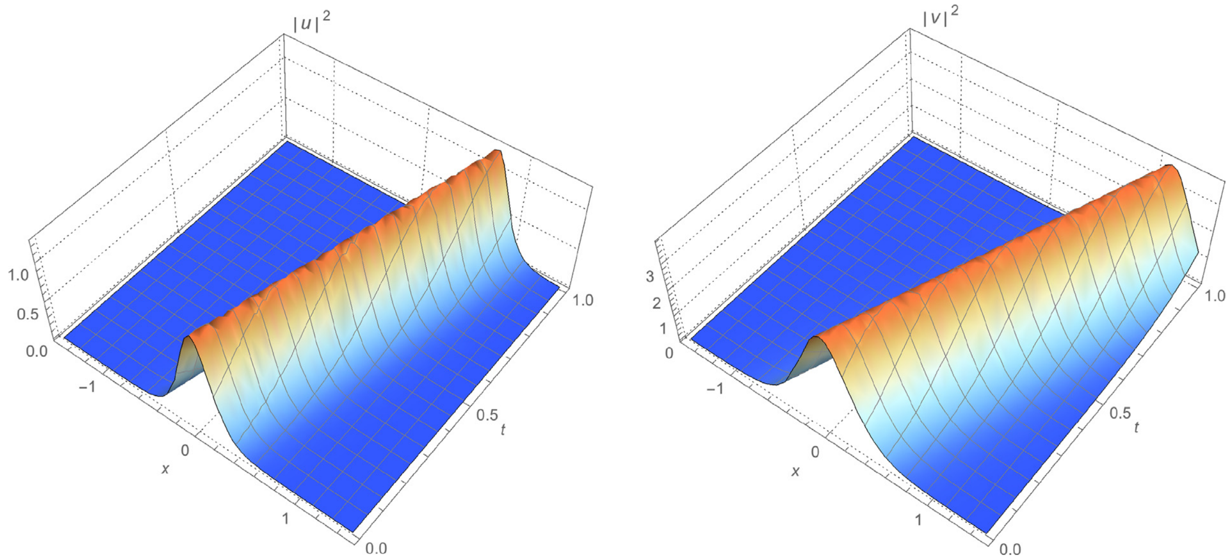

Fig. 1. Case-I, $n=0$, numerically computed profile of the bright solitons for Eq. (41).

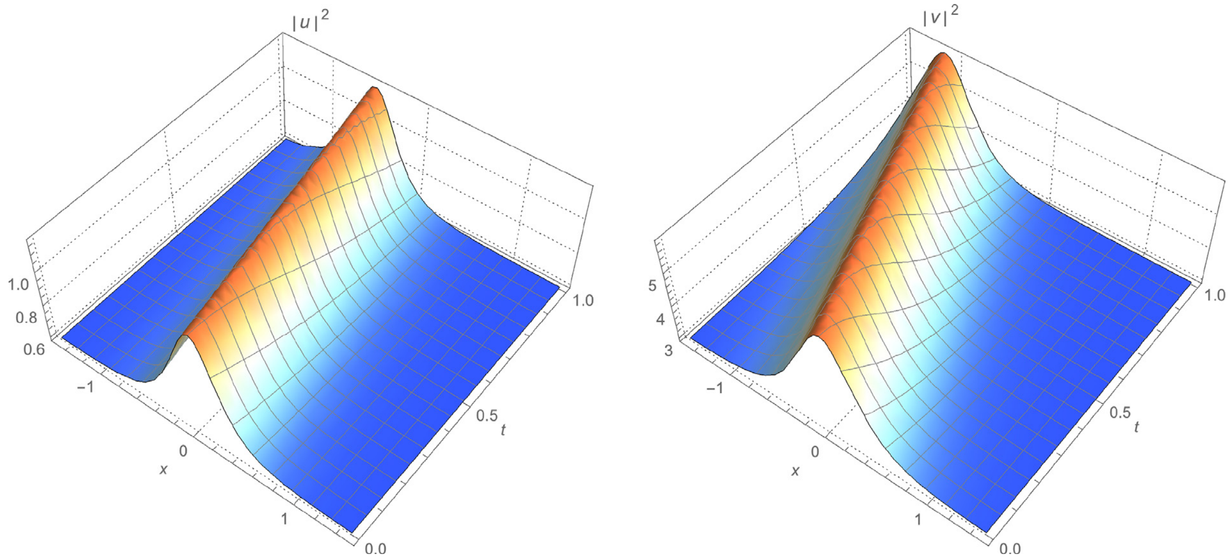

Fig. 2. Case-II, $n = 1$, numerically computed profile of the bright solitons for Eq. (82).

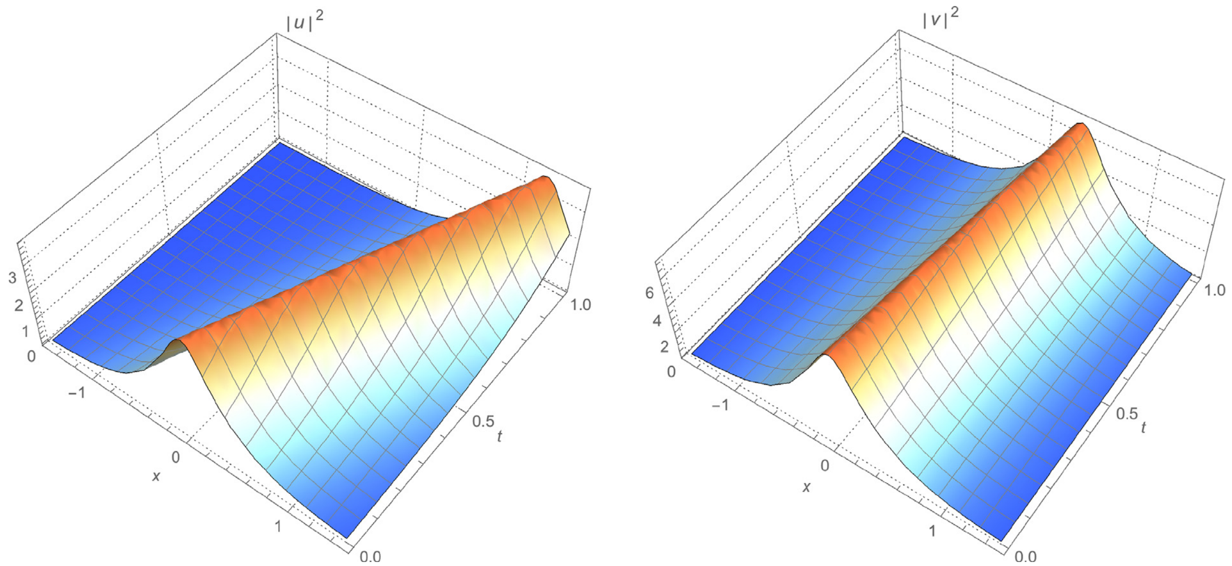

Fig. 3. Case-III, $n = 2$, numerically computed profile of the bright solitons for Eq. (123).

Table 1
Case-I, $n = 0$; coefficients of Eqs. (2) and (3) for bright and dark solitons.

l	a_l	b_l	c_l	d_l	f_l	α_l	β_l	η_l	ξ_l
1	1.2	0.01	0.024	0.50	4.20	0.33	1.20	0.01	0.32
2	1.5	0.07	0.028	0.25	3.20	0.16	1.10	0.03	0.12

Table 2
Case-II, $n = 1$; coefficients of Eqs. (4) and (5) for bright and dark solitons.

l	a_l	b_l	c_l	d_l	f_l	α_l	β_l	δ_l	γ_l	θ_l	η_l	ξ_l
1	1.0	0.09	0.013	0.12	1.00	0.08	1.09	− 0.02	0.7	1.6	1.00	1.34
2	1.6	0.04	0.044	0.80	1.20	0.10	1.30	− 0.01	0.3	1.3	1.60	3.01

Table 3
Case-III, $n = 2$; coefficients of Eqs. (6) and (7) for bright and dark solitons.

l	a_l	b_l	c_l	d_l	e_l	f_l	α_l	β_l	δ_l	γ_l	σ_l	η_l	r_l	ξ_l	θ_l
1	2.2	0.03	0.22	0.30	3.2	0.11	1.00	0.41	0.4	1.7	1.19	0.99	8.01	3.02	6.0
2	1.3	0.07	0.18	0.15	4.9	0.09	0.70	0.93	0.2	1.9	1.49	0.97	3.09	2.12	3.8

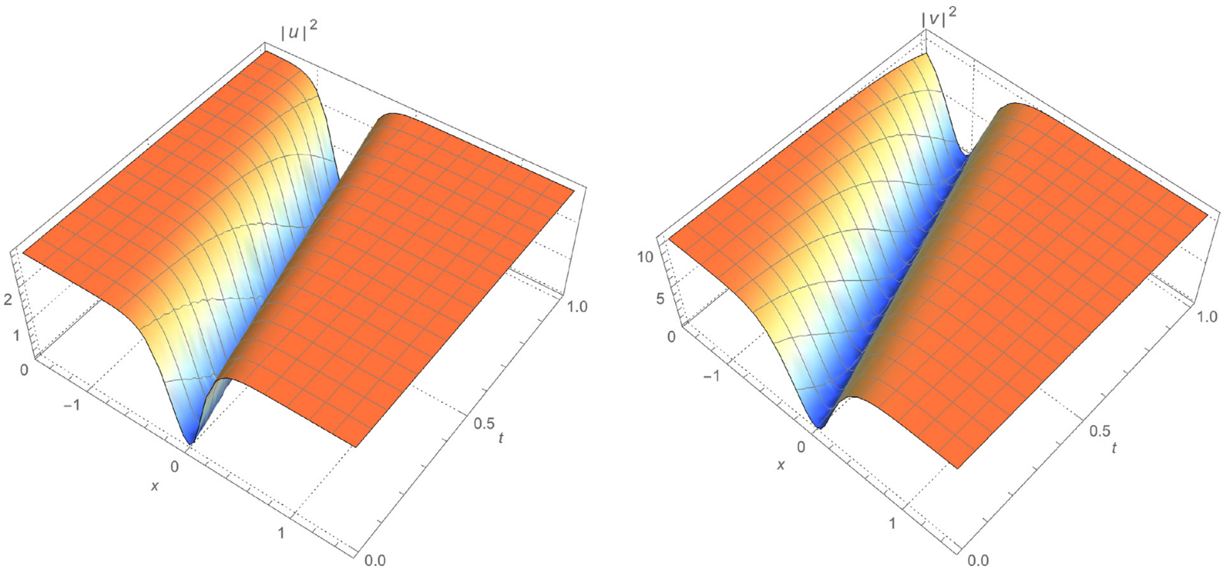

Fig. 4. Case-I, $n = 0$, numerically computed profile of the dark solitons for Eq. (36).

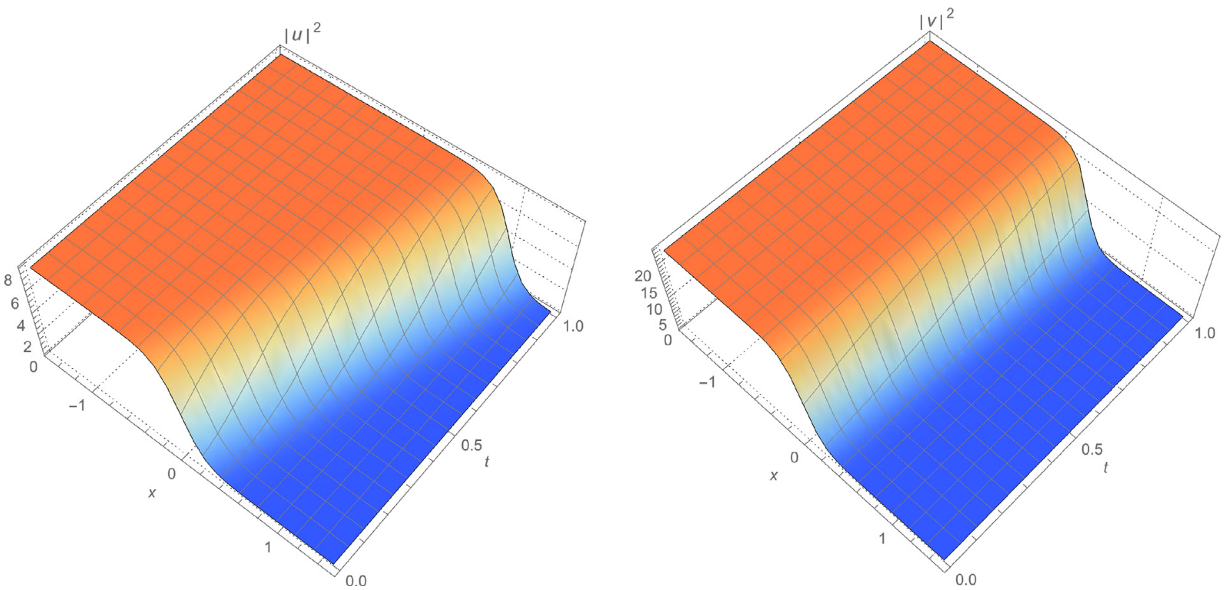

Fig. 5. Case-II, $n = 1$, numerically computed profile of the dark solitons for Eq. (75).

considered because the aim of this work is to locate soliton solutions. The general solutions [35–43] lead to a variety of solution formats that are not applicable in telecommunication industry. The results are thus very important in fibers with BGs. These results are important for future research activities. Later, additional nonlinear forms of refractive index will be taken into account. The soliton parameter dynamics will be looked for by the aid of variational principle, collective variables, moment method, soliton perturbation theory and several others. Additional integration methodologies will be employed in future to handle fiber BGs [21–30]. These are being currently worked upon and such results will be exhibited down the road.

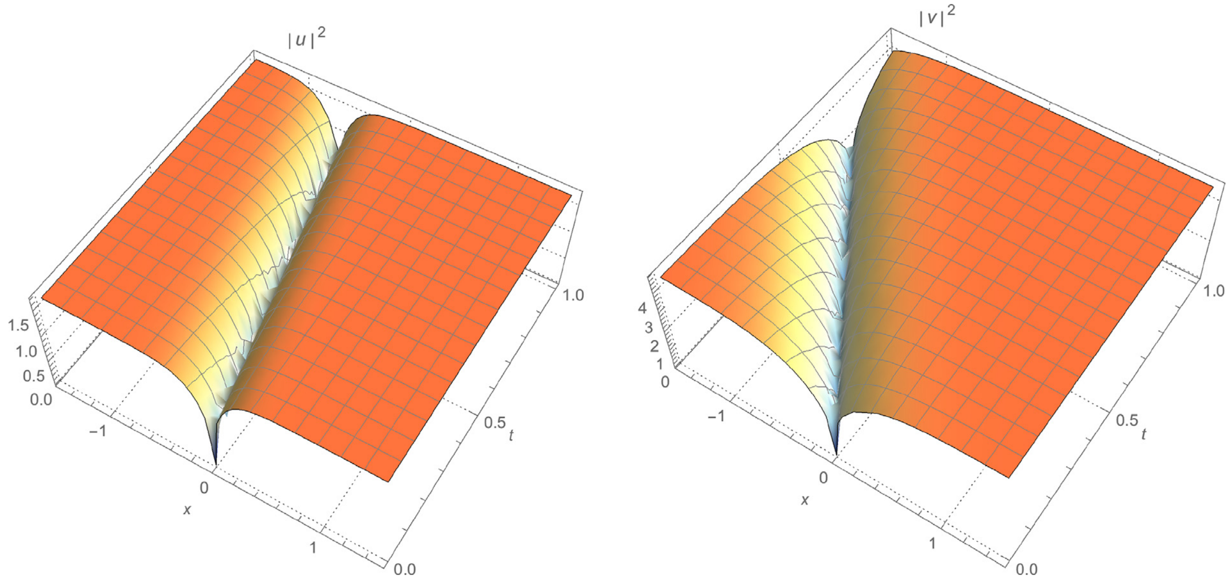

Fig. 6. Case-III, $n = 2$, numerically computed profile of the dark solitons for Eq. (118).

Declaration of Competing Interest

The authors also declare that there is no conflict of interest.

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