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# Optical solitons and conservation laws of Kudryashov's equation with improved modified extended tanh-function

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A B S T R A C T

This paper secures and enumerates soliton solutions to Kudryashov's equation. Bright, dark and singular optical soliton solutions have been listed for the model that are recovered by improved modified extended tanh-function approach. Finally, the conserved quantities are also exhibited.

## 1. Introduction

Optical solitons propagation dynamics have been modeled by a wide variety of models [1–13]. A few such models are the familiar nonlinear Schrödinger's equation (NLSE), Schrödinger–Hirota equation, Manakov equation, Sasa–Satsuma equation, Lakshmanan–Porsezian–Daniel equation, Fokas–Lennel's equation, Radhakrishnan–Kundu–Lakshmanan equation, Kundu–Mukherjee–Naskar equation, Gavitov–Turitsyn equation and several others. These models govern various kinds of soliton propagation and they include dispersive solitons, vector-coupled solitons, dispersion-managed solitons, optical dromions and other varieties. There are various additional occasions where NLSE is studied by Lie symmetry analysis along with several other models that include Boussinesq equation that models shallow water waves [14–19].

One of the latest innovative models that was proposed during 2019 is Kudryashov's equation (KE) [6]. What makes this model different is its structure of self-phase modulation (SPM). It is the only model where one has quadrupled power-law nonlinear effect. The only two pre-existing form of nonlinear models with more than one nonlinear term with power factor, that addressed soliton transmission dynamics, are NLSE with dual-power law nonlinearity and triple-power law nonlinearity. Thus, KE is the first model to address SPM with four nonlinear terms. The current paper therefore addresses KE by the aid of improved modified extended tanh-function

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method and this helps to retrieve bright, dark and singular solitons. Finally, the conservation laws are enumerated and it reveals the domain of the power-law nonlinearity parameter for which solitons exist. The details are enumerated in the rest of the paper, once KE is introduced and structured.

### 1.1. Governing model

KE in its dimensionless form is framed as [6]

$$\mathrm{i}q_t + aq_{xx} + \left(\frac{b_1}{|q|^{2n}} + \frac{b_2}{|q|^n} + b_3|q|^n + b_4|q|^{2n}\right)q = 0. \tag{1}$$

The first term in (1) stands for the linear temporal evolution, while $a$ gives the coefficient of group velocity dispersion (GVD). The terms $b_j$ for $1 \leq j \leq 4$ are nonlinear and stem from nonlinear refractive index of an optical fiber and give self-phase modulation (SPM) effect to the governing model. When $b_1 = b_2 = 0$, the model collapses to dual-power law of refractive index, while if $b_1 = b_2 = b_3 = 0$, one obtains power law. Kudryashov's model thus stands as an extension to these known forms of refractive indices. Lastly, the above cases with $n = 1$, are commonly referred to as parabolic law and the most fundamental Kerr law respectively.

We suppose a solution structure as

$$q(x,t) = P(\xi)e^{i\phi(x,t)}, \tag{2}$$

where

$$\xi = x - \mathrm{v}\ \mathrm{t}, \tag{3}$$

and $v$ is the velocity of the soliton. Next, phase factor is structured as:

$$\phi(x,t) = -\kappa x + \omega t + \theta, \tag{4}$$

where $\kappa$, $\omega$ and $\theta$ are the frequency, wave number and phase constant, respectively. Insert (2) into (1). Real and imaginary parts respectively yield

$$\mathrm{a}P^{2n}P'' + b_1P + b_2P^{n+1} + b_3P^{3n+1} + b_4P^{4n+1} - (\omega + a\kappa^2)P^{2n+1} = 0, \tag{5}$$

and

$$(v + 2a\kappa)P^{2n}P' = 0. \tag{6}$$

From (6), the speed is recovered as

$$v = -2a\kappa. \tag{7}$$

## 2. Revisitation of the algorithm

This section gives the description of the improved modified extended tanh-function method as below [9].

Consider nonlinear evolution equation with two independent variables $x$ and $t$:

$$F(u, u_t, u_x, u_{xx}, \ldots) = 0, \tag{8}$$

where $u = u(x, t)$ is an unknown function, $F$ is a polynomial in $u$ and its various partial derivatives $v_t, v_x$ with respect to $t$, $x$ respectively, in which the highest order derivatives and nonlinear terms are involved.

Step-1: Employ the wave transformation

$$u(x,t) = U(\xi), \quad \xi = k(x - \mathrm{c}\ \mathrm{t}), \tag{9}$$

where $k$, $c$ are constants that need to be established later. Then Eq. (8) can be transformed to the following nonlinear ordinary differential equation:

$$F(U, \mathrm{kc}U', k^2U'', \ldots) = 0. \tag{10}$$

Step-2: Suppose that the solution of Eq. (10) is taken to be

$$U(\xi) = \sum_{i=0}^{N} \alpha_i \varphi(\xi)^i + \sum_{i=1}^{N} \beta_i \varphi(\xi)^{-i}, \tag{11}$$

where $\varphi$ holds

$$\varphi' = \varepsilon\sqrt{a_0 + a_1\varphi + a_2\varphi^2 + a_3\varphi^3 + a_4\varphi^4}, \tag{12}$$

where $\varepsilon = \pm 1$. The last equation gives various kinds of fundamental solutions [1]. By virtue of these, the solutions to (8) can be revealed.

Step-3: Designate the positive integer number $N$ by balancing the nonlinear term and the highest order linear term in Eq. (10).

Step-4: Substitute the solution (11) which satisfies the condition (12) into Eq. (10). As a result of this substitution, one recovers a polynomial in $\varphi$. In this polynomial, one collects all terms of same powers and equating them to zero, one obtains an over-determined system of algebraic equations which can be handled by the Maple or Mathematica to find the unknown parameters $k$, $c$, $\alpha_0$, $\alpha_j$ and $\beta_j$ $(j = 1, 2, \ldots)$. As a consequence, one secures exact solutions to (8).

## 3. Application to KE

To reveal closed form solutions to the model, the transformation

$$P(\xi) = U(\xi)^{\frac{1}{n}}, \tag{13}$$

is utilized in Eq. (5) and thus

$$a\left(n\ UU'' + (1-n)U'^2\right) + b_1 n^2 + b_2 n^2 U + b_3 n^2 U^3 + b_4 n^2 U^4 - n^2\left(\omega + a\kappa^2\right)U^2 = 0. \tag{14}$$

Balancing $UU''$ with $U^4$ in Eq. (14) yields $N = 1$. Then Eq. (14) has the solution as

$$U(\xi) = \alpha_0 + \alpha_1\varphi(\xi) + \beta_1\varphi(\xi)^{-1}. \tag{15}$$

Substituting $U(\xi)$ and its derivatives with (12) into (14) and equating all the coefficients of $\varphi^j$, $j \in [-4, 4]$ to zero, then one derives the following system:

$\varphi^{-4}$ Coeff.:

$$\beta_1^2\left(aa_0(n+1) + b_4\beta_1^2 n^2\right) = 0, \tag{16}$$

$\varphi^{-3}$ Coeff.:

$$\frac{1}{2}\beta_1\left(\beta_1\left(aa_1(n+2) + 2\beta_1 n^2(4\alpha_0 b_4 + b_3)\right) + 4aa_0\alpha_0 n\right) = 0, \tag{17}$$

$\varphi^{-2}$ Coeff.:

$$\frac{1}{2}\beta_1\left(2\left(\beta_1\left(aa_2 - n^2\left(a\kappa^2 - 2b_4\left(2\alpha_1\beta_1 + 3\alpha_0^2\right) - 3\alpha_0 b_3 + \omega\right)\right) + 2aa_0\alpha_1(2n-1)\right) + 3aa_1\alpha_0 n\right) = 0, \tag{18}$$

$\varphi^{-1}$ Coeff.:

$$\begin{aligned}&\frac{1}{2}\beta_1\left(-4aa_1\alpha_1 + 2aa_3\beta_1 + 8aa_1\alpha_1 n - aa_3\beta_1 n + 6\alpha_1 b_3\beta_1 n^2 + 8\alpha_0^3 b_4 n^2 + 6\alpha_0^2 b_3 n^2 + 2b_2 n^2\right)\\&+\alpha_0\beta_1 n\left(aa_2 - 2n\left(a\kappa^2 - 6\alpha_1 b_4\beta_1 + \omega\right)\right) = 0,\end{aligned} \tag{19}$$

$\varphi^{0}$ Coeff.:

$$\begin{aligned}&n^2\left(-a\alpha_0^2\kappa^2 - \alpha_0^2\omega + \alpha_0^4 b_4 + \alpha_0^3 b_3 + \alpha_0 b_2 + b_1\right) + \frac{1}{2}\mathrm{a}\ \mathrm{n}(a_3\alpha_0\beta_1 + a_1\alpha_0\alpha_1) + aa_0\alpha_1^2(1-n)\\&-2\alpha_1\beta_1 n^2\left(a\kappa^2 + \omega\right) - 2aa_2\alpha_1\beta_1(1-2n) + aa_4\beta_1^2(1-n) + 6\alpha_1^2 b_4\beta_1^2 n^2 + 6\alpha_1\beta_1 n^2\left(2\alpha_0^2 b_4 + \alpha_0 b_3\right) = 0,\end{aligned} \tag{20}$$

$\varphi^{1}$ Coeff.:

$$\begin{aligned}&\frac{1}{2}\alpha_1\left(-4aa_3\beta_1 + aa_1\alpha_1(2-n) + 8aa_3\beta_1 n + 6\alpha_1 b_3\beta_1 n^2 + 8\alpha_0^3 b_4 n^2 + 6\alpha_0^2 b_3 n^2 + 2b_2 n^2\right)\\&+\alpha_1\alpha_0 n\left(aa_2 - 2n\left(a\kappa^2 - 6\alpha_1 b_4\beta_1 + \omega\right)\right) = 0,\end{aligned} \tag{21}$$

$\varphi^{2}$ Coeff.:

$$\frac{1}{2}\alpha_1\left(-2\alpha_1\left(n^2\left(a\kappa^2 - 6\alpha_0^2 b_4 - 3\alpha_0 b_3 + \omega\right) - aa_2\right) + 3aa_3\alpha_0 n + 4aa_4\beta_1(2n-1) + 8\alpha_1^2 b_4\beta_1 n^2\right) = 0, \tag{22}$$

$\varphi^{3}$ Coeff.:

$$\frac{1}{2}\alpha_1\left(\alpha_1\left(aa_3(n+2) + 2\alpha_1 n^2(4\alpha_0 b_4 + b_3)\right) + 4aa_4\alpha_0 n\right) = 0, \tag{23}$$

$\varphi^{4}$ Coeff.:

$$\alpha_1^2\left(aa_4(n+1)+\alpha_1^2 b_4 n^2\right)=0. \tag{24}$$

Handing this system and taking into account the solutions of (12), one gets the following results which brings about different types of solitons and other solutions

**Result-1**: If we set $a_0 = a_1 = a_3 = 0$, we obtain

$$\begin{aligned} &\alpha_0=-\frac{b_3(n+1)}{2b_4(n+2)},\quad \beta_1=0,\quad a_2=\frac{n^2\left(2b_4(n+2)^2\left(a\kappa^2+\omega\right)+3b_3^2(n+1)\right)}{2ab_4(n+2)^2},\\ &a_4=-\frac{\alpha_1^2 b_4 n^2}{\mathrm{a\ n}+a},\quad b_2=\frac{b_3\left(n^2-n-2\right)\left(b_4(n+2)^2\left(a\kappa^2+\omega\right)+b_3^2(n+1)\right)}{2b_4^2(n+2)^3},\\ &b_1=\frac{b_3^2(n-1)(n+1)^2\left(4b_4(n+2)^2\left(a\kappa^2+\omega\right)+5b_3^2(n+1)\right)}{16b_4^3(n+2)^4}. \end{aligned} \tag{25}$$

Then the corresponding solution of (14) is

$$q(x,t)=\left\{-\frac{b_3(n+1)}{2b_4(n+2)}+\sqrt{\frac{aa_2(n+1)}{b_4n^2}}\,\mathrm{sech}\left[\sqrt{\frac{n^2\left(2b_4(n+2)^2(a\kappa^2+\omega)+3b_3^2(n+1)\right)}{2ab_4(n+2)^2}}\,\xi\right]\right\}^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta)}. \tag{26}$$

This solution represents bright soliton. Provided that

$$ab_4\left\{2b_4(n+2)^2\left(a\kappa^2+\omega\right)+3b_3^2(n+1)\right\}\rangle 0.$$

Or

$$q(x,t)=\left\{-\frac{b_3(n+1)}{2b_4(n+2)}+\sqrt{\frac{aa_2(n+1)}{b_4n^2}}\,sec\left[\sqrt{-\frac{n^2\left(2b_4(n+2)^2(a\kappa^2+\omega)+3b_3^2(n+1)\right)}{2ab_4(n+2)^2}}\,\xi\right]\right\}^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta)}. \tag{27}$$

This solution represents singular periodic wave. Provided that

$$ab_4\left\{2b_4(n+2)^2\left(a\kappa^2+\omega\right)+3b_3^2(n+1)\right\}\langle 0.$$

**Result-2**: If we set $a_1 = a_3 = 0, a_0 = \frac{a_2^2}{4a_4}$, we obtain

Case-1:

$$\begin{aligned} &\alpha_0=-\frac{b_3(n+1)}{2b_4(n+2)},\quad a_2=\frac{n^2\left(\left(n+1\right)\left(a\kappa^2+\omega\right)+6\alpha_0^2b_4\right)}{a(n+1)},\quad \beta_1=0,\quad a_4=-\frac{\alpha_1^2b_4n^2}{a(n+1)},\\ &b_2=-\frac{\alpha_0(n-2)\left(\left(n+1\right)\left(a\kappa^2+\omega\right)+4\alpha_0^2b_4\right)}{n+1},\quad b_1=-\frac{(n-1)\left(\left(n+1\right)(a\kappa^2+\omega)+4\alpha_0^2b_4\right)^2}{4b_4(n+1)}. \end{aligned} \tag{28}$$

Then the corresponding solution of (14) is

$$q(x,t)=\left\{-\frac{b_3(n+1)}{2b_4(n+2)}\pm\sqrt{\frac{aa_2(n+1)}{2b_4n^2}}\,tanh\left[\sqrt{-\frac{n^2\left(2b_4(n+2)^2(a\kappa^2+\omega)+3b_3^2(n+1)\right)}{4ab_4(n+2)^2}}\,\xi\right]\right\}^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta)}. \tag{29}$$

This solution represents dark soliton solution. Provided that

$$ab_4\left\{2b_4(n+2)^2\left(a\kappa^2+\omega\right)+3b_3^2(n+1)\right\}\langle 0.$$

Or

$$q(x,t)=\left\{-\frac{b_3(n+1)}{2b_4(n+2)}\pm\sqrt{-\frac{aa_2(n+1)}{2b_4n^2}}\,tan\left[\sqrt{\frac{n^2\left(2b_4(n+2)^2(a\kappa^2+\omega)+3b_3^2(n+1)\right)}{4ab_4(n+2)^2}}\,\xi\right]\right\}^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta)}. \tag{30}$$

This solution represents singular periodic wave solution. Provided that

$$ab_4\left\{2b_4(n+2)^2\left(a\kappa^2+\omega\right)+3b_3^2(n+1)\right\}\rangle 0.$$

Case-2:

$$\alpha_0 = -\frac{b_3(n+1)}{2b_4(n+2)}, \quad \alpha_1 = 0, \quad a_2 = \frac{n^2\left(\left(n+1\right)\left(a\kappa^2+\omega\right)+6\alpha_0^2 b_4\right)}{a(n+1)}, \quad a_4 = -\frac{n^2\left(\left(n+1\right)\left(a\kappa^2+\omega\right)+6\alpha_0^2 b_4\right)^2}{4ab_4\beta_1^2(n+1)},$$
$$b_2 = -\frac{\alpha_0(n-2)\left(\left(n+1\right)\left(a\kappa^2+\omega\right)+4\alpha_0^2 b_4\right)}{n+1}, \quad b_1 = -\frac{(n-1)\left(\left(n+1\right)\left(a\kappa^2+\omega\right)+4\alpha_0^2 b_4\right)^2}{4b_4(n+1)}. \tag{31}$$

Then the corresponding solution of (14) is

$$q(x,t) = \left\{ -\frac{b_3(n+1)}{2b_4(n+2)} \pm \sqrt{\frac{aa_2(n+1)}{2b_4 n^2}} coth\left[\sqrt{-\frac{n^2\left(2b_4(n+2)^2(a\kappa^2+\omega)+3b_3^2(n+1)\right)}{4ab_4(n+2)^2}}\xi\right]\right\}^{\frac{1}{n}} e^{i(-\kappa x+\omega t+\theta)}. \tag{32}$$

This solution represents singular soliton solution. Provided that

$$ab_4\left\{2b_4(n+2)^2\left(a\kappa^2+\omega\right)+3b_3^2(n+1)\right\}\langle 0.$$

$$q(x,t) = \left\{ -\frac{b_3(n+1)}{2b_4(n+2)} \pm \sqrt{-\frac{aa_2(n+1)}{2b_4 n^2}} cot\left[\sqrt{\frac{n^2\left(2b_4(n+2)^2(a\kappa^2+\omega)+3b_3^2(n+1)\right)}{4ab_4(n+2)^2}}\xi\right]\right\}^{\frac{1}{n}} e^{i(-\kappa x+\omega t+\theta)}. \tag{33}$$

This solution represents singular periodic wave solution. Provided that

$$ab_4\left\{2b_4(n+2)^2\left(a\kappa^2+\omega\right)+3b_3^2(n+1)\right\}\rangle 0.$$

**Result-3**: If we set $a_1 = a_3 = 0, a_0 = \frac{a_2^2 m^2\left(1-m^2\right)}{a_4(2m^2-1)^2}$, we obtain

Case-1:

$$\alpha_0 = -\frac{b_3(n+1)}{2b_4(n+2)}, \quad \beta_1 = 0, \quad a_4 = -\frac{\alpha_1^2 b_4 n^2}{a(n+1)},$$
$$a_2 = \frac{n^2\left(2b_4(n+2)^2\left(a\kappa^2+\omega\right)+3b_3^2(n+1)\right)}{2ab_4(n+2)^2}, \quad b_2 = \frac{\alpha_0(n-2)\left(2\alpha_0^2 b_4 n^2 - aa_2(n+1)\right)}{n^2(n+1)},$$
$$b_1 = -\frac{(n-1)\left(-a^2a_2^2m^2\left(m^2-1\right)(n+1)^2 - aa_2\alpha_0^2 b_4\left(1-2m^2\right)^2 n^2(n+1) + \alpha_0^4 b_4^2\left(1-2m^2\right)^2 n^4\right)}{b_4\left(1-2m^2\right)^2 n^4(n+1)}. \tag{34}$$

Then the corresponding solution of (14) is

$$q(x,t) = \left\{ -\frac{b_3(n+1)}{2b_4(n+2)} + \sqrt{\frac{(n+1)aa_2m^2}{(2m^2-1)n^2b_4}}\ \mathrm{cn}\left[\sqrt{\frac{n^2\left(2b_4(n+2)^2(a\kappa^2+\omega)+3b_3^2(n+1)\right)}{2(2m^2-1)ab_4(n+2)^2}}\xi\right]\right\}^{\frac{1}{n}} e^{i(-\kappa x+\omega t+\theta)}. \tag{35}$$

If we choose $m = 1$, then we get

$$q(x,t) = \left\{ -\frac{b_3(n+1)}{2b_4(n+2)} + \sqrt{\frac{(n+1)aa_2}{n^2b_4}}\ \mathrm{sech}\left[\sqrt{\frac{n^2\left(2b_4(n+2)^2(a\kappa^2+\omega)+3b_3^2(n+1)\right)}{2ab_4(n+2)^2}}\xi\right]\right\}^{\frac{1}{n}} e^{i(-\kappa x+\omega t+\theta)}. \tag{36}$$

Case-2:

$$\begin{aligned}
&\alpha_0 = -\frac{b_3(n+1)}{2b_4(n+2)}, \quad \alpha_1 = 0, \quad a_4 = \frac{aa_2^2 m^2\left(m^2-1\right)(n+1)}{b_4\beta_1^2\left(2m^2-1\right)^2 n^2},\\
&\omega = \frac{-6\alpha_0^2 b_4 n\sqrt{am^2\left(m^2-1\right)(n+1)} - a\beta_1\left(2m^2-1\right)(n+1)\sqrt{a_4 b_4} - a\kappa^2 n(n+1)\sqrt{am^2\left(m^2-1\right)(n+1)}}{n(n+1)\sqrt{am^2\left(m^2-1\right)(n+1)}},\\
&b_2 = \frac{\alpha_0\sqrt{b_4}(n-2)\left(2\alpha_0^2\sqrt{b_4}\,n\sqrt{am^2\left(m^2-1\right)(n+1)} + a\sqrt{a_4}\,\beta_1\left(2m^2-1\right)(n+1)\right)}{n(n+1)\sqrt{am^2\left(m^2-1\right)(n+1)}},\\
&b_1 = -\frac{(n-1)\left(\alpha_0^4 b_4 n^2\sqrt{am^2\left(m^2-1\right)(n+1)} + a\sqrt{a_4}\,\alpha_0^2\sqrt{b_4}\,\beta_1\left(2m^2-1\right)n(n+1) - aa_4\beta_1^2(n+1)\sqrt{am^2\left(m^2-1\right)(n+1)}\right)}{n^2(n+1)\sqrt{am^2\left(m^2-1\right)(n+1)}}.
\end{aligned} \tag{37}$$

Then the corresponding solution of (14) is

$$\begin{aligned}
q(x,t) &= \left\{-\frac{b_3(n+1)}{2b_4(n+2)} + \sqrt{-\frac{(2m^2-1)(m^2-1)(n+1)aa_2}{b_4(2m^2-1)^2 n^2}}\,\mathrm{nc}\left[\sqrt{\frac{a_2}{2m^2-1}}\xi\right]\right\}^{\frac{1}{n}}\\
&\times e^{i\left(-\kappa x + \left\{\frac{-6\alpha_0^2 b_4 n\sqrt{am^2\left(m^2-1\right)(n+1)} - a\beta_1\left(2m^2-1\right)(n+1)\sqrt{a_4 b_4} - a\kappa^2 n(n+1)\sqrt{am^2\left(m^2-1\right)(n+1)}}{n(n+1)\sqrt{am^2\left(m^2-1\right)(n+1)}}\right\}t + \theta\right)}.
\end{aligned} \tag{38}$$

This solution represents Jacobi elliptic function solution.

**Result-4**: If we set $a_1 = a_3 = 0, a_0 = \frac{a_2^2\left(1-m^2\right)}{a_4\left(2-m^2\right)^2}$ we obtain

Case-1:

$$\begin{aligned}
&\alpha_0 = -\frac{b_3(n+1)}{2b_4(n+2)}, \quad \beta_1 = 0, \quad a_4 = -\frac{\alpha_1^2 b_4 n^2}{a(n+1)},\\
&a_2 = \frac{n^2\left(2b_4(n+2)^2\left(a\kappa^2+\omega\right) + 3b_3^2(n+1)\right)}{2ab_4(n+2)^2}, \quad b_2 = \frac{\alpha_0(n-2)\left(2\alpha_0^2 b_4 n^2 - aa_2(n+1)\right)}{n^2(n+1)},\\
&b_1 = -\frac{(n-1)\left(-a^2a_2^2\left(m^2-1\right)(n+1)^2 - aa_2\alpha_0^2 b_4\left(m^2-2\right)^2 n^2(n+1) + \alpha_0^4 b_4^2\left(m^2-2\right)^2 n^4\right)}{b_4\left(m^2-2\right)^2 n^4(n+1)}.
\end{aligned} \tag{39}$$

Then the corresponding solution of (14) is

$$q(x,t) = \left\{-\frac{b_3(n+1)}{2b_4(n+2)} + \sqrt{\frac{(n+1)aa_2 m^2}{(2-m^2)n^2 b_4}}\,\mathrm{dn}\left[\sqrt{\frac{n^2\left(2b_4(n+2)^2(a\kappa^2+\omega) + 3b_3^2(n+1)\right)}{2(2-m^2)ab_4(n+2)^2}}\xi\right]\right\}^{\frac{1}{n}} e^{i(-\kappa x+\omega t+\theta)}. \tag{40}$$

If we choose $m = 1$, then we get

$$q(x,t) = \left\{-\frac{b_3(n+1)}{2b_4(n+2)} + \sqrt{\frac{(n+1)aa_2}{n^2 b_4}}\,\mathrm{sech}\left[\sqrt{\frac{n^2\left(2b_4(n+2)^2(a\kappa^2+\omega) + 3b_3^2(n+1)\right)}{2ab_4(n+2)^2}}\xi\right]\right\}^{\frac{1}{n}} e^{i(-\kappa x+\omega t+\theta)}. \tag{41}$$

Case-2:

$$
\begin{gathered}
\alpha_0=-\frac{b_3(n+1)}{2b_4(n+2)},\quad \alpha_1=0,\quad a_4=\frac{aa_2^2m^2\left(m^2-1\right)(n+1)}{b_4\beta_1^2\left(m^2-2\right)^2n^2},\\
\omega=\frac{-6\alpha_0^2b_4n\sqrt{am^2\left(m^2-1\right)(n+1)}-a\beta_1\left(m^2-2\right)(n+1)\sqrt{a_4b_4}-a\kappa^2n(n+1)\sqrt{am^2\left(m^2-1\right)(n+1)}}{n(n+1)\sqrt{am^2\left(m^2-1\right)(n+1)}},\\
b_2=\frac{\alpha_0\sqrt{b_4}(n-2)\left(2\alpha_0^2\sqrt{b_4}\,n\sqrt{am^2\left(m^2-1\right)(n+1)}+a\sqrt{a_4}\,\beta_1\left(m^2-2\right)(n+1)\right)}{n(n+1)\sqrt{am^2\left(m^2-1\right)(n+1)}},\\
b_1=-\frac{(n-1)\left(\alpha_0^4b_4n^2\sqrt{am^2\left(m^2-1\right)(n+1)}+a\sqrt{a_4}\,\alpha_0^2\sqrt{b_4}\,\beta_1\left(m^2-2\right)n(n+1)-aa_4\beta_1^2(n+1)\sqrt{am^2\left(m^2-1\right)(n+1)}\right)}{n^2(n+1)\sqrt{am^2\left(m^2-1\right)(n+1)}}.
\end{gathered}
\tag{42}
$$

Then the corresponding solution of (14) is

$$
\begin{aligned}
q(x,t)&=\left\{-\frac{b_3(n+1)}{2b_4(n+2)}+\sqrt{-\frac{(m^2-2)(m^2-1)(n+1)aa_2}{b_4(m^2-2)^2n^2}}\ \mathrm{nd}\left[\sqrt{\frac{a_2}{m^2-2}}\xi\right]\right\}^{\frac{1}{n}}\\
&\quad\times e^{i\left(-\kappa x+\left\{\frac{-6\alpha_0^2b_4n\sqrt{am^2\left(m^2-1\right)(n+1)}-a\beta_1\left(m^2-2\right)(n+1)\sqrt{a_4b_4}-a\kappa^2n(n+1)\sqrt{am^2\left(m^2-1\right)(n+1)}}{n(n+1)\sqrt{am^2\left(m^2-1\right)(n+1)}}\right\}t+\theta\right)}.
\end{aligned}
\tag{43}
$$

This solution represents Jacobi elliptic function solution.

**Result-5**: If we set $a_1=a_3=0, a_0=\frac{a_2^2m^2}{a_4(m^2+1)^2}$ we obtain

Case-1:

$$
\begin{gathered}
\alpha_0=-\frac{b_3(n+1)}{2b_4(n+2)},\quad \beta_1=0,\quad a_4=-\frac{\alpha_1^2b_4n^2}{a(n+1)},\\
a_2=\frac{n^2\left(2b_4(n+2)^2\left(a\kappa^2+\omega\right)+3b_3^2(n+1)\right)}{2ab_4(n+2)^2},\quad b_2=\frac{\alpha_0(n-2)\left(2\alpha_0^2b_4n^2-aa_2(n+1)\right)}{n^2(n+1)},\\
b_1=-\frac{(n-1)\left(-a^2a_2^2\left(m^2-1\right)(n+1)^2-aa_2\alpha_0^2b_4\left(m^2+1\right)^2n^2(n+1)+\alpha_0^4b_4^2\left(m^2+1\right)^2n^4\right)}{b_4\left(m^2-2\right)^2n^4(n+1)}.
\end{gathered}
\tag{44}
$$

Then the corresponding solution of (14) is

$$
q(x,t)=\left\{-\frac{b_3(n+1)}{2b_4(n+2)}+\sqrt{\frac{(n+1)aa_2m^2}{(m^2+1)n^2b_4}}\ \mathrm{sn}\left[\sqrt{-\frac{n^2\left(2b_4(n+2)^2(a\kappa^2+\omega)+3b_3^2(n+1)\right)}{2(m^2+1)ab_4(n+2)^2}}\xi\right]\right\}^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta)}.
\tag{45}
$$

If we choose $m=1$, then we get

$$
q(x,t)=\left\{-\frac{b_3(n+1)}{2b_4(n+2)}+\sqrt{\frac{(n+1)aa_2}{2n^2b_4}}\,tanh\left[\sqrt{-\frac{n^2\left(2b_4(n+2)^2(a\kappa^2+\omega)+3b_3^2(n+1)\right)}{4ab_4(n+2)^2}}\xi\right]\right\}^{\frac{1}{n}}e^{i(-\kappa x+\omega t+\theta)}.
\tag{46}
$$

Case-2:

$$\alpha_0=-\frac{b_3(n+1)}{2b_4(n+2)},\quad \alpha_1=0,\quad a_4=\frac{aa_2^2m^2\left(m^2-1\right)(n+1)}{b_4\beta_1^2\left(m^2+1\right)^2n^2},$$
$$\omega=\frac{-6\alpha_0^2b_4n\sqrt{am^2\left(m^2-1\right)(n+1)}-a\beta_1\left(m^2+1\right)(n+1)\sqrt{a_4b_4}-a\kappa^2n(n+1)\sqrt{am^2\left(m^2-1\right)(n+1)}}{n(n+1)\sqrt{am^2\left(m^2-1\right)(n+1)}},$$
$$b_2=\frac{\alpha_0\sqrt{b_4}(n-2)\left(2\alpha_0^2\sqrt{b_4}\,n\sqrt{am^2\left(m^2-1\right)(n+1)}+a\sqrt{a_4}\,\beta_1\left(m^2+1\right)(n+1)\right)}{n(n+1)\sqrt{am^2\left(m^2-1\right)(n+1)}}, \tag{47}$$
$$b_1=-\frac{(n-1)\left(\alpha_0^4b_4n^2\sqrt{am^2\left(m^2-1\right)(n+1)}+a\sqrt{a_4}\,\alpha_0^2\sqrt{b_4}\,\beta_1\left(m^2+1\right)n(n+1)-aa_4\beta_1^2(n+1)\sqrt{am^2\left(m^2-1\right)(n+1)}\right)}{n^2(n+1)\sqrt{am^2\left(m^2-1\right)(n+1)}}.$$

Then the corresponding solution of (14) is

$$q(x,t)=\left\{-\frac{b_3(n+1)}{2b_4(n+2)}+\sqrt{-\frac{(m^2+1)(m^2-1)(n+1)aa_2}{b_4(m^2+1)^2n^2}}\,\mathrm{ns}\left[\sqrt{\frac{a_2}{m^2-2}}\xi\right]\right\}^{\frac{1}{n}}\times e^{i\left(-\kappa x+\left\{\frac{-6\alpha_0^2b_4n\sqrt{am^2(m^2-1)(n+1)}-a\beta_1\left(m^2+1\right)(n+1)\sqrt{a_4b_4}-a\kappa^2n(n+1)\sqrt{am^2(m^2-1)(n+1)}}{n(n+1)\sqrt{am^2(m^2-1)(n+1)}}\right\}t+\theta\right)}. \tag{48}$$

This solution represents Jacobi elliptic function solution, where $a_2>0$.

**Result-6**: If we set $a_2=a_4=0$, we obtain

$$\alpha_1=0,\quad a_0=-\frac{b_4\beta_1^2n^2}{\mathrm{a\ n}+a},\quad a_1=-\frac{2\beta_1n^2(2\alpha_0b_4(n+2)+b_3(n+1))}{a(n^2+3n+2)},$$
$$a_3=-\frac{n^2\left(b_1\left(n^2+3n+2\right)-\alpha_0^3(n-1)(3\alpha_0b_4(n+2)+2b_3(n+1))\right)}{a\alpha_0\beta_1(n-1)\left(n^2+3n+2\right)},$$
$$b_2=-\frac{(n-2)\left(b_1\left(n^2+3n+2\right)+\alpha_0^3(n-1)(5\alpha_0b_4(n+2)+4b_3(n+1))\right)}{2\alpha_0(n-1)\left(n^2+3n+2\right)}, \tag{49}$$
$$\omega=\frac{6\alpha_0b_3(n+1)-(n+2)\left(a\kappa^2(n+1)-6\alpha_0^2b_4\right)}{n^2+3n+2}.$$

Then the corresponding solution of (14) is

$$q(x,t)=\left\{\alpha_0+\frac{\beta_1}{\wp\left(\sqrt{-\frac{n^2\left(b_1(n^2+3n+2)-\alpha_0^3(n-1)(3\alpha_0b_4(n+2)+2b_3(n+1))\right)}{4a\alpha_0\beta_1(n-1)(n^2+3n+2)}}\xi,g_2,g_3\right)}\right\}^{\frac{1}{n}}\times e^{i\left(-\kappa x+\left\{\frac{6\alpha_0b_3(n+1)-(n+2)\left(a\kappa^2(n+1)-6\alpha_0^2b_4\right)}{n^2+3n+2}\right\}t+\theta\right)}. \tag{50}$$

This solution represents Weierstrass elliptic doubly periodic solution, where

$$g_2=-\frac{8\alpha_0\beta_1^2(n-1)(2\alpha_0b_4(n+2)+b_3(n+1))}{b_1(n^2+3n+2)-\alpha_0^3(n-1)(3\alpha_0b_4(n+2)+2b_3(n+1))},$$

$$g_3=\frac{4\alpha_0b_4\beta_1^3(n-1)(n+2)}{\alpha_0^3(n-1)(3\alpha_0b_4(n+2)+2b_3(n+1))-b_1(n^2+3n+2)}.$$

**Result-7**: If we set $a_0=a_1=a_2=0$, we obtain

$$
\begin{aligned}
&\beta_1 = 0, \quad a_3 = -\frac{2\alpha_1 n^2 (2\alpha_0 b_4 (n+2) + b_3 (n+1))}{a(n^2+3n+2)}, \quad a_4 = -\frac{\alpha_1^2 b_4 n^2}{\mathrm{a\ n} + a}, \\
&b_1 = \frac{\alpha_0^3 (n-1)(3\alpha_0 b_4 (n+2) + 2b_3 (n+1))}{n^2+3n+2}, \\
&b_2 = -\frac{\alpha_0^2 (n-2)(4\alpha_0 b_4 (n+2) + 3b_3 (n+1))}{n^2+3n+2}, \\
&\omega = \frac{6\alpha_0 b_3 (n+1) - (n+2)\left(a\kappa^2 (n+1) - 6\alpha_0^2 b_4\right)}{n^2+3n+2}.
\end{aligned}
\tag{51}
$$

Then the corresponding solution of (14) is

$$
q(x,t) = \left\{ \alpha_0 - \frac{\frac{8\alpha_1 n^2 (2\alpha_0 b_4 (n+2) + b_3 (n+1))}{a(n^2+3n+2)}}{\frac{4{\alpha_1}^2 n^4 (2\alpha_0 b_4 (n+2) + b_3 (n+1))^2}{a^2 (n^2+3n+2)^2}\xi^2 + \frac{4\alpha_1^2 b_4 n^2}{a(n+1)}} \right\}^{\frac{1}{n}} e^{i\left(-\kappa x + \left(\frac{6\alpha_0 b_3 (n+1) - (n+2)\left(a\kappa^2 (n+1) - 6\alpha_0^2 b_4\right)}{n^2+3n+2}\right)t + \theta\right)}, \tag{52}
$$

or

$$
q(x,t) = \left\{ \alpha_0 + \frac{2\alpha_0 b_4 (n+2) + b_3 (n+1)}{b_4 (n+2)} e^{\mp \frac{n(2\alpha_0 b_4 (n+2) + b_3 (n+1))}{(n+2)\sqrt{ab_4 (n+1)}}\xi} \right\}^{\frac{1}{n}} e^{i\left(-\kappa x + \left(\frac{6\alpha_0 b_3 (n+1) - (n+2)\left(a\kappa^2 (n+1) - 6\alpha_0^2 b_4\right)}{n^2+3n+2}\right)t + \theta\right)}, \tag{53}
$$

where $ab_4 > 0$. These solutions represent rational and exponential solutions.

**Result-8**: If we set $a_0 = \frac{a_1^2}{4a_2}, a_3 = a_4 = 0$, we obtain

$$
\begin{aligned}
&\alpha_1 = 0, \quad b_1 = \frac{a\alpha_0^2 (n-1)(a_1 \alpha_0 - 2a_2 \beta_1)^2}{4a_2 \beta_1^2 n^2}, \\
&b_2 = -\frac{a\alpha_0 (n-2)\left(-3a_1 a_2 \alpha_0 \beta_1 + a_1^2 \alpha_0^2 + 2a_2^2 \beta_1^2\right)}{2a_2 \beta_1^2 n^2}, \\
&a_1 = -\frac{2\beta_1 n^2 (2\alpha_0 b_4 (n+2) + b_3 (n+1))}{a(n^2+3n+2)}, \\
&a_2 = -\frac{n^2 (2\alpha_0 b_4 (n+2) + b_3 (n+1))^2}{ab_4 (n+1)(n+2)^2}, \\
&\omega = -a\kappa^2 - \frac{3aa_1 \alpha_0}{\beta_1 n^2} + \frac{3aa_1^2 \alpha_0^2}{2a_2 \beta_1^2 n^2} + \frac{aa_2}{n^2}.
\end{aligned}
\tag{54}
$$

Then the corresponding solution of (14) is

$$
\begin{aligned}
q(x,t) \quad &= \left\{ \alpha_0 + \frac{\beta_1}{-\frac{b_4 \beta_1 (n+1)(n+2)^2}{(n^2+3n+2)(2\alpha_0 b_4 (n+2) + b_3 (n+1))} + e^{\pm\sqrt{-\frac{n^2 (2\alpha_0 b_4 (n+2) + b_3 (n+1))^2}{ab_4 (n+1)(n+2)^2}}\xi}} \right\}^{\frac{1}{n}} \\
&\times e^{i\left(-\kappa x + \left\{-a\kappa^2 - \frac{3aa_1 \alpha_0}{\beta_1 n^2} + \frac{3aa_1^2 \alpha_0^2}{2a_2 \beta_1^2 n^2} + \frac{aa_2}{n^2}\right\}t + \theta\right)},
\end{aligned}
\tag{55}
$$

where $ab_4 < 0$.

**Result-9**: If we set $a_1 = a_3 = a_4 = 0$, we obtain

$$\begin{aligned} &\alpha_0 = -\frac{b_3(n+1)}{2b_4(n+2)}, \quad \alpha_1 = 0, \quad a_0 = -\frac{b_4\beta_1^2 n^2}{\mathrm{a\ n} + a}, \\ &a_2 = \frac{n^2\left(2b_4(n+2)^2\left(a\kappa^2+\omega\right)+3b_3^2(n+1)\right)}{2ab_4(n+2)^2}, \\ &b_1 = \frac{b_3^2(n-1)(n+1)^2\left(4b_4(n+2)^2\left(a\kappa^2+\omega\right)+5b_3^2(n+1)\right)}{16b_4^3(n+2)^4}, \\ &b_2 = \frac{b_3\left(n^2-n-2\right)\left(b_4(n+2)^2\left(a\kappa^2+\omega\right)+b_3^2(n+1)\right)}{2b_4^2(n+2)^3}. \end{aligned} \tag{56}$$

Then the corresponding solution of (14) is

$$q(x,t) = \left\{ -\frac{b_3(n+1)}{2b_4(n+2)} \pm \frac{\sqrt{\frac{(\mathrm{a\ n}+a)\left(2b_4(n+2)^2(a\kappa^2+\omega)+3b_3^2(n+1)\right)}{2ab_4^2(n+2)^2}}}{sin\left[\sqrt{-\frac{n^2\left(2b_4(n+2)^2(a\kappa^2+\omega)+3b_3^2(n+1)\right)}{2ab_4(n+2)^2}}\,\xi\right]} \right\}^{\frac{1}{n}} e^{i(-\kappa x+\omega t+\theta)}, \tag{57}$$

where

$$\mathrm{ab}_4 < 0, 2b_4(n+2)^2\left(a\kappa^2+\omega\right)+3b_3^2(n+1) > 0.$$

Or

$$q(x,t) = \left\{ -\frac{b_3(n+1)}{2b_4(n+2)} \pm \frac{\sqrt{-\frac{(\mathrm{a\ n}+a)\left(2b_4(n+2)^2(a\kappa^2+\omega)+3b_3^2(n+1)\right)}{2ab_4^2(n+2)^2}}}{sinh\left[\sqrt{\frac{n^2\left(2b_4(n+2)^2(a\kappa^2+\omega)+3b_3^2(n+1)\right)}{2ab_4(n+2)^2}}\,\xi\right]} \right\}^{\frac{1}{n}} e^{i(-\kappa x+\omega t+\theta)}, \tag{58}$$

where

$$\mathrm{ab}_4 > 0, 2b_4(n+2)^2\left(a\kappa^2+\omega\right)+3b_3^2(n+1) < 0.$$

These solutions represent periodic wave solution and hyperbolic wave solution.

**Result-10**: If we set $a_0 = a_1 = 0$, we obtain:

Case-1:

$$\begin{aligned} &\beta_1 = 0, \quad a_3 = -\frac{2\alpha_1 n^2(2\alpha_0 b_4(n+2)+b_3(n+1))}{a(n^2+3n+2)}, \quad a_4 = -\frac{\alpha_1^2 b_4 n^2}{\mathrm{a\ n}+a}, \\ &a_2 = \frac{n^2\left(\left(n+2\right)\left(\left(n+1\right)\left(a\kappa^2+\omega\right)-6\alpha_0^2 b_4\right)-6\alpha_0 b_3(n+1)\right)}{a(n^2+3n+2)}, \\ &b_1 = \frac{\alpha_0^2(n-1)\left(\left(n+2\right)\left(\left(n+1\right)\left(a\kappa^2+\omega\right)-3\alpha_0^2 b_4\right)-4\alpha_0 b_3(n+1)\right)}{n^2+3n+2}, \\ &b_2 = -\frac{\alpha_0(n-2)\left(\left(n+2\right)\left(\left(n+1\right)\left(a\kappa^2+\omega\right)-2\alpha_0^2 b_4\right)-3\alpha_0 b_3(n+1)\right)}{n^2+3n+2}. \end{aligned} \tag{59}$$

Then the corresponding solution of (14) is

$$q(x,t) = \left\{ \alpha_0 + \frac{\alpha_1 a_2\, \mathrm{sech}^2\left[\frac{1}{2}\sqrt{a_2}\,\xi\right]}{\pm 2\sqrt{-\frac{\alpha_1^2 b_4 n^4\left(\left(n+2\right)\left(\left(n+1\right)(a\kappa^2+\omega)-6\alpha_0^2 b_4\right)-6\alpha_0 b_3(n+1)\right)}{a^2(n+1)^2(n+2)}}\, tanh\left[\frac{1}{2}\sqrt{a_2}\,\xi\right] - a_3} \right\}^{\frac{1}{n}} e^{i(-\kappa x+\omega t+\theta)}, \tag{60}$$

or

$$q(x,t) = \left\{ \alpha_0 - \frac{\alpha_1 a_2 sec^2\left[\frac{1}{2}\sqrt{-a_2}\,\xi\right]}{\pm 2\sqrt{\frac{\alpha_1^2 b_4 n^4\left((n+2)\left((n+1)(a\kappa^2+\omega)-6\alpha_0^2 b_4\right)-6\alpha_0 b_3(n+1)\right)}{a^2(n+1)^2(n+2)}}\, tan\left[\frac{1}{2}\sqrt{-a_2}\,\xi\right] + a_3} \right\}^{\frac{1}{n}} e^{i(-\kappa x+\omega t+\theta)}, \tag{61}$$

where $a_2 < 0$ and $a_2$, $a_3$ are given by (59).

Case-2:

$$\begin{aligned}
&\beta_1 = 0, \quad b_1 = \frac{a\alpha_0^2(n-1)(a_3\alpha_1 - 2a_4\alpha_0)^2}{4a_4\alpha_1^2 n^2}, \quad a_4 = -\frac{\alpha_1^2 b_4 n^2}{a(n+1)},\\
&b_2 = -\frac{a\alpha_0(n-2)\left(8a_4^2\alpha_0^2 - 6a_3a_4\alpha_1\alpha_0 + a_3^2\alpha_1^2\right)}{4a_4\alpha_1^2 n^2},\\
&a_2 = -\frac{n^2(2\alpha_0 b_4(n+2) + b_3(n+1))^2}{ab_4(n+1)(n+2)^2},\\
&\omega = -a\kappa^2 - \frac{3aa_3\alpha_0}{\alpha_1 n^2} + \frac{6aa_4\alpha_0^2}{\alpha_1^2 n^2} + \frac{aa_3^2}{4a_4 n^2}.
\end{aligned} \tag{62}$$

Then the corresponding solution of (14) is

$$\begin{aligned}
q(x,t) \ &= \left\{ \alpha_0 \pm \sqrt{\frac{(2\alpha_0 b_4(n+2) + b_3(n+1))^2}{4\alpha_1^2 b_4^2 (n+2)^2}} \left(1 + tanh\left[\sqrt{-\frac{n^2(2\alpha_0 b_4(n+2)+b_3(n+1))^2}{4ab_4(n+1)(n+2)^2}}\,\xi\right]\right)\right\}^{\frac{1}{n}}\\
&\times e^{i\left(-\kappa x + \left\{-a\kappa^2 - \frac{3aa_3\alpha_0}{\alpha_1 n^2} + \frac{6aa_4\alpha_0^2}{\alpha_1^2 n^2} + \frac{aa_3^2}{4a_4 n^2}\right\} t + \theta\right)},
\end{aligned} \tag{63}$$

where

$$a_3 = \pm 2\sqrt{\frac{\alpha_1^2 n^4 (2\alpha_0 b_4(n+2) + b_3(n+1))^2}{a^2(n+1)^2(n+2)^2}}, \tag{64}$$

and

$$ab_4 < 0. \tag{65}$$

## 4. Conservation laws

Three conserved quantities that naturally emerge from KE are power ($P$), linear momentum ($M$) and Hamiltonian ($H$). These are listed here. The general structure of the conservation laws for NLSE, with arbitrary functional form of SPM, has been discussed and reported earlier [20]. Typically, such an arbitrary form of SPM leads to these three conserved quantities. But, first, the bright 1-soliton solution to KE (1) is restructured as:

$$q(x,t) = A\,\mathrm{sech}^{\frac{1}{n}}[B(x-\mathrm{ct})]e^{i(-\kappa x+\omega t+\theta_0)}. \tag{66}$$

The three conserved quantities consequently emerge as [20]:

$$P = \int_{-\infty}^{\infty} |q|^2 \mathrm{dx} = \frac{A^2}{B}\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}, \tag{67}$$

$$M = \mathrm{i}a\int_{-\infty}^{\infty}\left(\mathrm{qq}_x^* - q^* q_x\right)\mathrm{dx} = \frac{a\kappa A^2}{B}\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}, \tag{68}$$

and

$$
\begin{aligned}
H &= \int_{-\infty}^{\infty}\left(a|q_x|^2 - \frac{b_1}{1-n}|q|^{2-2n} - \frac{2b_2}{2-n}|q|^{2-n} - \frac{2b_3}{n+2}|q|^{n+2} - \frac{b_4}{n+1}|q|^{2n+2}\right)\mathrm{dx} \\
&= \frac{\mathrm{aA}^2}{n^2(n+2)B}\left\{(n+2)\left(B^2+n^2\kappa^2\right)-2B^2\right\}\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)} - \frac{b_1A^{2-2n}}{(1-n)B}\frac{\Gamma\left(\frac{1}{n}-1\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)} \\
&\quad - \frac{2b_2A^{2-n}}{(2-n)B}\frac{\Gamma\left(\frac{1}{n}-\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}\right)} - \frac{2\mathrm{nb}_3A^{n+2}}{(n+2)B}\frac{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}\right)} - \frac{2b_4A^{2n+2}}{(n+1)(n+2)B}\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}.
\end{aligned}
\tag{69}
$$

Thus, from the Hamiltonian one can easily conclude by looking at the Euler's gamma functions that solitons for KE would exist for

$$0 < n < 1. \tag{70}$$

## 5. Conclusions

This paper secured and listed bright, dark and singular optical soliton solutions to KE that carried four forms of nonlinear terms each having a power-law parameter. The conserved quantities are subsequently computed and enumerated. The structure of these conservation laws provided the range of values of the power-law nonlinearity parameter for which the soliton solutions would exist. The paper thus provides a comprehensive piece of information to the KE model.

These results form a foundation stone to further future ventures with KE. Some such avenues are exploring the model in birefringent fibers, DWDM topology, studying the model in presence of both deterministic as well as stochastic perturbation terms. This would yield quasi-monochromatic optical soliton dynamics as well as locating the mean-free velocity of the soliton with stochasticity. Integration of the model in presence of deterministic Hamiltonian perturbation terms is also on the agenda. Thus, a lot is up for grabs!

## Conflict of interest

The authors also declare that there is no conflict of interest.

## Declaration of Competing Interest

The authors report no declarations of interest.

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